Coupling Strength Allocation for Synchronization in Complex Networks Using Spectral Graph Theory

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***Abstract*—Using spectral graph theory and especially its graph comparison techniques, we propose new methodologies to allocate coupling strengths to guarantee global complete synchronization in complex networks. The key step is that all the eigenvalues of the Laplacian matrix associated with a given network can be estimated by utilizing flexibly topological features of the network. The proposed methodologies enable the construction of different coupling-strength combinations in response to different knowledge about subnetworks. Adaptive allocation strategies can be carried out as well using only local network topological information. Besides formal analysis, we use simulation examples to demonstrate how to apply the methodologies to typical complex networks.**

***Index Terms*—Complex networks, coupling strength allocation, spectral graph theory, synchronization.**

I. Introduction

SYNCHRONIZATION phenomena in various complex networks have attracted great attention in the past decades [2]–[9]. This research area has many diverse applications, such as flocking behavior in birds and social insects, social dynamics in populations, and coordination strategies in mobile autonomous robots. Several significant systematic approaches have been proposed in this research field. The master stability function method has been established as a powerful tool in [2] to study the local synchronization problem for linearly coupled chaotic systems. In [5], a general systematic framework was presented for the study of synchronization of nonlinear dynamical systems with diffusive couplings. When deriving various synchronization conditions, one focus is how to assign coupling strengths to the interconnections between systems, and intuitively it is natural to assume that the synchronized behavior of any two systems is always possible to take place provided that the coupling strength between them is sufficiently large [5]. In order to provide a lower bound on the coupling strengths for interconnected systems, for a network with diffusive and symmetric couplings, one can further investigate the synchronizability of the network by examining the magnitude of the second smallest eigenvalue of the Laplacian matrix, called algebraic connectivity, of the network [10]. A range of related research problems, such as robustness issues, have been studied following this approach [11]–[13]. In parallel, a different line of research has also been developed to study the global synchronization of complex networks, which uses extensively the topological information of the graph that describes the couplings between the systems in a network [14]. The main idea is to construct a bound on the total length of all the paths passing through a chosen edge in the graph. This bound can then be exploited to allocate coupling strengths to all the edges in order to achieve global synchronization in the network. One aim of this paper is to bridge the main results developed separately in [5] and [14] with these two different approaches and propose new coupling strength allocation methods to guarantee global complete synchronization in complex networks. Newly obtained results from spectral graph theory will be utilized toward this end.

Graph comparison techniques have been developed in the past to bound the second smallest eigenvalues of Laplacian matrices of *undirected* graphs [15]–[17], where the bounds are obtained by embedding complete graphs into the graph under study. More general ideas for graph comparison have been reported in [18], [19], where the comparison of combinatorial features can be carried out between two arbitrary graphs with the same vertex set for the purpose of bounding any eigenvalues of Laplacian matrices of the graphs. In this paper, we follow the approach delineated in [18], [19] to study conditions based on graph comparison for synchronization in complex networks. By doing so, we prove that the synchronization condition given in [14] for allocating coupling strengths can be explained by comparing the network graph with the corresponding complete graph. We then propose different coupling strength allocation strategies by comparing the network graphs with other typical network structures. Since adaptive allocations can be carried out using only local network topological information, our method is especially useful in large, time-varying, complex networks. So the main contribution of the paper lies in a set of new methodologies using graph comparison to allocate coupling strengths to guarantee complete synchronization in complex networks.

The rest of the paper is organized as follows. In Section II, we review a classical complex dynamical network model and some relevant results in the literature on synchronization. Using tools from spectral graph theory, we study conditions based

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on graph comparison for synchronization in complex dynamical networks and propose new methods to allocate coupling strengths, in Sections III and IV. In Section V we look into synchronizability of networks using graph comparison tools developed in the previous sections. Section VI provides numerical simulations on network synchronization. Finally, conclusions are given in Section VII.

II. Problem Setup and Preliminaries

We consider a network of $n > 1$ coupled identical oscillators whose dynamics are described by

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{n} \varepsilon_{ij}(t) P x_j, \qquad i = 1, \ldots, n \tag{1}$$

where $x_i \in \mathbb{R}^d$ is the state of the ith oscillator, $f(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ denotes the identical self-dynamics of each oscillator, $\varepsilon_{ij}(t) \geq 0 (i \neq j)$ describes the time-varying strength of the coupling from oscillator j to i at time t, $\varepsilon_{ii}(t) = -\sum_{j=1, j\neq i}^{n} \varepsilon_{ij}(t)$, and the diagonal (0,1)-matrix $P \in \mathbb{R}^{d\times d}$ determines through which components of the states that the oscillators are coupled together. We assume that the couplings between oscillators are symmetric, namely $\varepsilon_{ij}(t) = \varepsilon_{ji}(t)$. The couplings between the oscillators can be conveniently described by a weighted graph $\mathbb{G}(t) = (\mathcal{V}, \mathcal{E}, \varepsilon(t))$ with the vertex set $\mathcal{V} = \{1, \ldots, n\}$, the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weight function $\varepsilon : \mathcal{E} \to \mathbb{R}$. There is an edge between vertices i and j if and only if $\varepsilon_{ij}(t) = \varepsilon_{ji}(t) > 0$ and the weights ε_{ij} may change with time. Let $L_{\mathbb{G}(t)} \in \mathbb{R}^{n\times n}$ be the Laplacian matrix [20] of the graph $\mathbb{G}(t)$. Then the ijth entry of $L_{\mathbb{G}(t)}$ is $-\varepsilon_{ij}(t)$ for $1 \leq i, j \leq n$.

System (1) has been used widely to study under what conditions the coupled oscillators can achieve asymptotically global and complete synchronization, where for any initial condition, $\|x_i(t) - x_j(t)\| \to 0$ as $t \to \infty$ for all i, j [5]. In this paper, we explore such synchronization conditions using spectral graph theory. Toward this end, we make one standard technical assumption about system (1).

Assumption 1: For a sufficiently large positive constant a, it holds that

$$(x_j - x_i)^T \left[(f(x_j) - f(x_i)) - aP(x_j - x_i)\right] \leq -c\|x_j - x_i\|^2 \tag{2}$$

for some $c > 0$ and for any $x_i, x_j \in \mathbb{R}^d$.

Here the constant a is determined by both the function f and the inner coupling matrix P. Assumption 1 implies that two coupled oscillators are always able to get synchronized when their coupling is sufficiently strong. An equivalent assumption has been made in [14], [21], and [22], which guarantee that the whole network of oscillators can get synchronized when the coupling strengths between oscillators are sufficiently large. Explanations on the equivalence of the two assumptions are given in the Appendix. Such networks that satisfy Assumption 1 include most of the coupled limit-cycle or chaotic oscillators. For those networks that do not satisfy this assumption, it is likely that increasing the coupling strengths between some pairs of oscillators may destroy the network's locally stable synchronous states [2]. We refer the interested reader to [2] and a more recent paper [23] for a systematic classification of different network synchronization behavior.

In the following, we introduce a general synchronization criterion for networks with time-varying dynamics. For a matrix $A \in \mathbb{R}^{n\times n}$, we say $A \succ 0$ (respectively, $A \succeq 0$) if $x^T A x$ is positive (respectively, non-negative) for all nonzero $x \in \mathbb{R}^n$. We use $\mathcal{W}_s$ to denote the set of irreducible, symmetric matrices that have zero row sums and non-positive off-diagonal elements.

Lemma 1 (Minor Rephrasing of Theorem 2 from [10] and a Result in Chapter 4 From [5]): Let $Y(t)$ be a d-by-d time-varying matrix and V a d-by-d symmetric, positive definite matrix such that $(y - z)^T V(f(y, t) + Y(t)y - f(z, t) - Y(t)z) \leq -c\|y - z\|^2$ for some $c > 0$ and all y, z, t. Then system (1) synchronizes globally if there exists an n-by-n matrix $U \in \mathcal{W}_s$ such that

$$(U \otimes V)\left(L_{\mathbb{G}(t)} \otimes (-P) - I_n \otimes Y(t)\right) \preceq 0 \tag{3}$$

for all t, where $\otimes$ denotes the Kronecker product [24].

Now we present a synchronization criterion using properties of graphs.

Theorem 1: Under Assumption 1, the synchronization manifold of system (1) is globally asymptotically stable if there exists a connected undirected graph $\mathbb{G}_0$ with the same vertex set of $\mathbb{G}(t)$ such that

$$L_{\mathbb{G}_0} L_{\mathbb{G}(t)} - aL_{\mathbb{G}_0} \succ 0, \quad \text{for all } t. \tag{4}$$

Proof: Assumption 1 on the self-dynamics $f(\cdot)$ is equivalent to the condition that $(y - z)^T V(f(y, t) + Y(t)y - f(z, t) - Y(t)z) \leq -c\|y - z\|^2$ when we set $Y(t) = -aP$, $V = I_d$. To apply Lemma 1, we choose $Y(t) = -aP$, $V = I_d$, and $U = L_{\mathbb{G}_0}$. Then from (3) we have $(L_{\mathbb{G}_0} \otimes I_d)(L_{\mathbb{G}(t)} \otimes (-P) - I_n \otimes (-aP)) \preceq 0$, i.e., $L_{\mathbb{G}_0} L_{\mathbb{G}(t)} \otimes (-P) - aL_{\mathbb{G}_0} \otimes (-P) \preceq 0$. Since $-P \preceq 0$, this is satisfied if $L_{\mathbb{G}_0} L_{\mathbb{G}(t)} - aL_{\mathbb{G}_0} \succeq 0$. Therefore, the complete synchronization of system (1) is guaranteed if $L_{\mathbb{G}_0} L_{\mathbb{G}(t)} \succ aL_{\mathbb{G}_0}$ for all t. ■

III. Synchronization Criteria Using Graph Comparison With Complete Graphs

In this section, we look at graphical synchronization criteria for undirected complex networks. Toward this end, we introduce some notations and discuss some algebraic properties of graphs. We say $A \succ B$ if $A - B \succ 0$. Similarly, we say $A \succeq B$ if $A - B \succeq 0$. We extend this notation for graphs as follows.

Definition 1: For two undirected graphs $\mathbb{G}$ and $\mathbb{H}$ with the same vertex set $\mathcal{V} = \{1, \ldots, n\}$, we say

$$\mathbb{G} \succeq \mathbb{H}$$

if their Laplacian matrices satisfy $L_{\mathbb{G}} \succeq L_{\mathbb{H}}$.

For a graph $\mathbb{G}$ with vertex set $\mathcal{V}$, we use λ_k, $1 \leq k \leq n$, to denote the kth smallest eigenvalue of $L_{\mathbb{G}}$. For graphs $\mathbb{G}$ and $\mathbb{H}$ with the same vertex set, we consider some multiple $c\mathbb{G}$ of graph $\mathbb{G}$. Using Courant–Fischer Theorem [19], one can easily prove the following result.

Lemma 2: If $\mathbb{G}$ and $\mathbb{H}$ are the graphs with the same vertex set $\mathcal{V}$ satisfying $c\mathbb{G} \succeq \mathbb{H}$, then

$$c\lambda_k(\mathbb{G}) \geq \lambda_k(\mathbb{H})$$

for all $1 \leq k \leq n$.

Theorem 1 gives a synchronization condition based on graph comparison. One natural idea is to compare the system graph $\mathbb{G}(t)$ with the complete graph. Let $\mathbb{K}_n$ denote the unweighted, undirected complete graph [20] with n vertices. If we take the graph $\mathbb{G}_0$ in Theorem 1 to be $\mathbb{K}_n$, then one has that the synchronization manifold of system (1) is globally asymptotically stable if $L_{\mathbb{K}_n} L_{\mathbb{G}(t)} \succ a L_{\mathbb{K}_n}$ for all t. Note that $L_{\mathbb{K}_n} = nI_n - J$ where J is the n-by-n all-one matrix. We know then $L_{\mathbb{K}_n} L_{\mathbb{G}(t)} = (nI_n - J)L_{\mathbb{G}(t)} = nL_{\mathbb{G}(t)} \succ aL_{\mathbb{K}_n}$. Thus, the synchronization manifold of system (1) is globally asymptotically stable if $L_{\mathbb{G}(t)} \succ (a/n)L_{\mathbb{K}_n}$. So we have arrived at the following theorem.

Theorem 2: Suppose that graph $\mathbb{G}(t)$ is undirected and connected. Under Assumption 1, the synchronization manifold of system (1) is globally asymptotically stable if

$$\mathbb{G}(t) \succ \frac{a}{n}\mathbb{K}_n, \quad \text{for all } t. \tag{5}$$

The implication of Theorem 2 is profound. For any coupled oscillators whose couplings are described by a weighted undirected graph $\mathbb{G}(t)$, one can always examine whether $\mathbb{G}(t) \succ (a/n)\mathbb{K}_n$ holds by comparing $\mathbb{G}(t)$ to the complete graph with identical edge weight a/n. Now we show that the inequality in Theorem 2 can be stated differently.

Theorem 3: For an undirected graph $\mathbb{G}(t)$, it holds that

$$\mathbb{G}(t) \succ \frac{a}{n}\mathbb{K}_n \Leftrightarrow \lambda_2(\mathbb{G}(t)) > a.$$

Proof: "$\Rightarrow$": From $\mathbb{G}(t) \succ (a/n)\mathbb{K}_n$ for each t and Lemma 2, we know $\lambda_2(\mathbb{G}(t)) > (a/n)\lambda_2(\mathbb{K}_n)$ for each t. Since $\lambda_2(\mathbb{K}_n) = n$, it then must be true that $\lambda_2(\mathbb{G}(t)) > a$ for each t.

"$\Leftarrow$": Since the all-one vector $\mathbf{1} = [1, \ldots, 1]^T$ is in the kernel of $L_{\mathbb{G}(t)}$ and $L_{\mathbb{K}_n}$, to prove $\mathbb{G}(t) \succ (a/n)\mathbb{K}_n$, it suffices to prove that $x^T(L_{\mathbb{G}(t)} - (a/n)L_{\mathbb{K}_n})x > 0$ for any $x \in \mathbb{R}^n$ that is not in the kernel of $L_{\mathbb{G}(t)}$ and $L_{\mathbb{K}_n}$. Furthermore, one can easily see that it suffices to prove that $x^T(L_{\mathbb{G}(t)} - (a/n)L_{\mathbb{K}_n})x > 0$ for all the vector $x \in \mathbb{R}^n$ orthogonal to $\mathbf{1}$.

For any vector x orthogonal to $\mathbf{1}$, from Courant–Fischer theorem [19], one has

$$\lambda_2(\mathbb{G}(t)) \le \min_{x \perp \mathbf{1}} \frac{x^T L_{\mathbb{G}(t)} x}{x^T x}, \quad \text{for each } t.$$

Thus one has

$$\lambda_2(\mathbb{G}(t))\, x^T x \le x^T L_{\mathbb{G}(t)} x, \quad \forall x \perp \mathbf{1}.$$

Since $\lambda_2(\mathbb{G}(t)) > a$, we know $x^T L_{\mathbb{G}(t)} x > a x^T x$ for all $x \perp \mathbf{1}$. Using the fact that $L_{\mathbb{K}_n} = nI_n - J$, we have

$$\begin{aligned} x^T\left(nL_{\mathbb{G}(t)} - aL_{\mathbb{K}_n}\right)x &= x^T\left(nL_{\mathbb{G}(t)} - a(nI_n - J)\right)x \\ &= nx^T L_{\mathbb{G}(t)} x - nax^T x + ax^T J x \\ &\ge n\left(x^T L_{\mathbb{G}(t)} x - ax^T x\right) > 0 \end{aligned}$$

which implies that $n\mathbb{G}(t) - a\mathbb{K}_n \succ 0$ for all $x \perp \mathbf{1}$, namely $\mathbb{G}(t) \succ (a/n)\mathbb{K}_n$. ■

Remark 1: In Theorem 3 in [10], a lower bound for $\lambda_2(\mathbb{G}(t))$ has been given to guarantee the synchronization of coupled dynamical oscillators under certain assumptions. In Theorem 3, we have shown the equivalence between graph comparisons and bounding from below the second smallest eigenvalues of the Laplacian matrices of graphs.

To apply more tools from spectral graph theory, we need to introduce another equivalent definition of the Laplacian matrix of graphs. Following [18], we define the elementary Laplacian $L_{(u,v)}$ to be the Laplacian of the graph containing just the edge of unit weight between vertices u and v. Then for an undirected graph $\mathbb{G}(t) = (\mathcal{V}, \mathcal{E}, \varepsilon(t))$ consisting of the vertex set $\mathcal{V}$, the edge set $\mathcal{E}$, and the weight function $\varepsilon : \mathcal{E} \to \mathbb{R}$, its Laplacian matrix has the form

$$L(\mathbb{G}(t)) \triangleq \sum_{(u,v)\in\mathcal{E}} \varepsilon_{(u,v)}(t) \cdot L_{(u,v)}. \tag{6}$$

Moreover, we say graph $\mathbb{G}$ is unweighted if the weights $\varepsilon_{(u,v)} = 1$ for all $u \ne v$.

Now we introduce two graphical inequalities, which have been proved in [18].

Lemma 3 [18]: Let $c_1, \ldots, c_{n-1} > 0$. It holds that

$$c\left(\sum_{i=1}^{n-1} c_i L_{(i,i+1)}\right) \succeq L_{(1,n)}$$

where $c = \sum_i (1/c_i)$.

If we take $c_1 = c_2 = \ldots = c_{n-1} = 1$, then Lemma 3 becomes the following result.

Lemma 4 [18]: It holds that

$$(n-1)\left(\sum_{i=1}^{n-1} L_{(i,i+1)}\right) \succeq L_{(1,n)}.$$

With these graphical tools at hand, Theorem 2 can be further used to give graphical conditions for the synchronization of system (1). In the following, we present some sufficient conditions for synchronization using features of graph $\mathbb{G}(t)$. Consider a set of paths $\mathcal{P} = \{\mathcal{P}_{ij} | i, j = 1, \ldots, n, j > i\}$, one for each pair of distinct vertices i and j. We denote the length of the path $\mathcal{P}_{ij}$ by $|\mathcal{P}_{ij}|$, which is the number of edges in $\mathcal{P}_{ij}$. We assume that there are altogether m edges in the edge set $\mathcal{E}$ of graph $\mathbb{G}(t)$. If we label the edges of $\mathbb{G}(t)$ by $1, \ldots, m$, then the lower bounds on the coupling strengths of all the edges can be constructed to guarantee that the inequality in Theorem 2 holds. We state it more formally as follows.

Theorem 4: Suppose that graph $\mathbb{G}(t)$ is undirected and connected. Under Assumption 1, the synchronization manifold of system (1) is globally asymptotically stable if

$$\varepsilon_k(t) > \frac{b_k}{n} a, \qquad \text{for } k = 1, \ldots, m \text{ and for all } t$$

where $b_k = \sum_{j>i; k\in\mathcal{P}_{ij}} |\mathcal{P}_{ij}|$ is the sum of the lengths of all those paths $\mathcal{P}_{ij}$ in $\mathcal{P}$ that contain edge k.

Proof: From the definition introduced by (6), it holds that

$$\frac{a}{n} L_{\mathbb{K}_n} = \frac{a}{n} \sum_{i=1}^{n-1} \sum_{j>i} L_{(i,j)}.$$

For each pair of (i, j) where $j > i$, we choose one path $\mathcal{P}_{ij}$ in $\mathbb{G}$ that connects i and j. Then one can apply Lemma 4 by comparing the sum of all the Laplacian matrices L_k, $k \in \mathcal{P}_{ij}$, of all the edges along this chosen path and the Laplacian matrix $L_{(i,j)}$ of the single edge (i, j), which leads to

$$|\mathcal{P}_{ij}| \sum_{k \in \mathcal{P}_{ij}} L_k \succeq L_{(i,j)}. \tag{7}$$

Choosing such paths in the topological graph $\mathbb{G}$ for all the pairs of i, j where $j > i$, one obtains that

$$\begin{aligned} \frac{a}{n} L_{\mathbb{K}_n} &\preceq \frac{a}{n} \sum_{j>i} \left(|\mathcal{P}_{ij}| \sum_{k \in \mathcal{P}_{ij}} L_k \right) \\ &= \frac{a}{n} \sum_{k=1}^{m} \left(\sum_{\substack{j>i \\ k \in \mathcal{P}_{ij}}} |\mathcal{P}_{ij}| \right) L_k \\ &= \frac{a}{n} \sum_{k=1}^{m} b_k L_k \\ &\prec \sum_{k=1}^{m} \varepsilon_k(t) L_k = L_{\mathbb{G}(t)} \end{aligned}$$

where $b_k = \sum_{\substack{j>i \\ k \in \mathcal{P}_{ij}}} |\mathcal{P}_{ij}|$ has been defined in Theorem 4. And the last inequality holds trivially when $\varepsilon_k(t) > (a/n) b_k$ for each edge k. Therefore, the constructed coupling strengths ε_k for $k = 1, \ldots, m$ guarantee that $\mathbb{G}(t) \succ (a/n)\mathbb{K}_n$ holds. Thus we arrive at the conclusion. ■

Remark 2: Theorem 4 presents a synchronization condition for allocating coupling strengths for $\mathbb{G}(t)$. The same result has been obtained in [14]. Here we give a different interpretation of the result and prove it by using combinatorial features of the topological graph $\mathbb{G}$, which leads to the construction of efficient algorithms determining the coupling strengths as we will show later. In addition, our allocation method using graph comparison is much easier to implement in applications.

Up to now, we have only compared graph $\mathbb{G}(t)$ with the complete graph $\mathbb{K}_n$. It is natural to ask what different synchronization criteria can be obtained if we compare $\mathbb{G}(t)$ with other graphs. We explore in this direction in the next section.

IV. Synchronization Criteria Using Graph Comparison With Other Typical Graphs

A. Coupling Strength Allocation

In Theorem 2, the synchronization criteria are given based on the comparison between the given graph $\mathbb{G}(t)$ with the complete graph. In addition, Theorem 4 gives lower bounds of coupling strengths in order to achieve complete synchronization. In what follows, we show how to allocate coupling strengths systematically by comparing $\mathbb{G}(t)$ with some typical graphs. We list below some results about the eigenvalues of some typical graphs.

Lemma 5 [18]: (a) The Laplacian matrix of the complete graph $\mathbb{K}_n$ has eigenvalue 0 with multiplicity 1 and n with multiplicity $n - 1$.

(b) The Laplacian matrix of the ring graph $\mathbb{R}_n$ has eigenvalues $2 - 2\cos(2\pi k/n)$ for $0 \le k \le n/2$.

(c) The Laplacian matrix of the path graph $\mathbb{P}_n$ has eigenvalues $2 - 2\cos(\pi k/n)$ for $0 \le k \le n - 1$.

(d) The Laplacian matrix of the star graph $\mathbb{S}_n$ has 1 as its second smallest eigenvalue.

In fact, one can compare any two undirected and connected graphs, and obtain a graphical inequality as a result. One can find more details in [18] and [19]. Thus, for a graph whose second smallest eigenvalue is known, we can always compare it with $\mathbb{G}(t)$ and obtain a set of coupling strengths to guarantee complete synchronization of the dynamical network (1). To show how to implement this idea, we give an example now on comparing graph $\mathbb{G}(t)$ with a star graph. Similar results can be achieved when $\mathbb{G}(t)$ is compared with other typical graphs, such as ring graphs, path graphs, and any graphs with known second smallest eigenvalues.

Now consider an n-vertex star graph $\mathbb{S}_n$, in which without loss of generality we assume vertex 1 has $n-1$ neighbors. Then $L_{\mathbb{S}_n} = \sum_{i=2}^{n} L_{(1,i)}$. We consider two cases for all the edges $(1, i)$, $2 \le i \le n$, in $\mathbb{S}_n$.

1) Edge $(1, i)$ is not in the edge set $\mathcal{E}$ of $\mathbb{G}(t)$. Since $\mathbb{G}(t)$ is connected, there must exist some paths in $\mathbb{G}(t)$ connecting vertices 1 and i. We choose arbitrarily one of those paths, which is denoted by $\mathcal{P}_{1,i}$. Then we have

$$L_{(1,i)} \preceq |\mathcal{P}_{1,i}| \sum_{k \in \mathcal{P}_{1,i}} L_k. \tag{8}$$

2) Edge $(1, i)$ is in $\mathcal{E}$. There are two options: one is to use $(1, i)$ directly and the other is to choose arbitrarily another path between vertices 1 and i, if such a path exists. We set the probability of the first option to be $1 - \alpha_i$, and that for the second α_i where $0 \le \alpha_i < 1$. If there are no paths between 1 and i other than the edge $(1, i)$, we always set $\alpha_i = 0$. Thus we have

$$L_{(1,i)} \preceq (1 - \alpha_i) L_{(1,i)} + \alpha_i |\mathcal{P}_{1,i}| \sum_{k \in \mathcal{P}_{1,i}} L_k. \tag{9}$$

Note that (8) is the special case of (9) when α_i is taken to be 1. Hence, we can use the inequality (9) with a proper choice of $\alpha_i \in [0, 1]$ for each $i \in \{2, \ldots, n\}$, and so

$$\begin{aligned} L_{\mathbb{S}_n} &= \sum_{i=2}^{n} L_{(1,i)} \\ &\preceq \sum_{i=2}^{n} \left[(1 - \alpha_i) L_{(1,i)} + \alpha_i |\mathcal{P}_{1,i}| \sum_{k \in \mathcal{P}_{1,i}} L_k \right] \\ &= \sum_{i=2}^{n} (1 - \alpha_i) L_{(1,i)} + \sum_{i=2}^{n} \left[\alpha_i |\mathcal{P}_{1,i}| \sum_{k \in \mathcal{P}_{1,i}} L_k \right] \\ &= \sum_{k=1}^{m} \left[\sum_{\substack{i=2 \\ k \in \mathcal{P}_{1,i}}}^{n} \alpha_i |\mathcal{P}_{1,i}| \right] L_k + \sum_{i=2}^{n} (1 - \alpha_i) L_{(1,i)} \\ &= \sum_{k=1}^{m} \left[\sum_{\substack{i=2 \\ k \in \mathcal{P}_{1,i}}}^{n} \alpha_i |\mathcal{P}_{1,i}| + \varphi(1 - \alpha_i) \right] L_k \end{aligned}$$

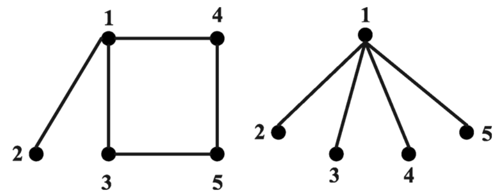

Fig. 1. Comparing $\mathbb{G}_5$ with the star $\mathbb{S}_5$.

where the real valued function $\varphi(1-\alpha_i)$ satisfies

$$\varphi(1-\alpha_i)=\begin{cases}1-\alpha_i\neq 0 & \text{if } (1,i) \text{ is the edge } k,\\ 0 & \text{otherwise.}\end{cases}$$

Let

$$b'_k=\sum_{\substack{i=2\\ k\in\mathcal{P}_{1,i}}}^{n}\alpha_i|\mathcal{P}_{1,i}|+\varphi(1-\alpha_i). \tag{10}$$

Then we have $\mathbb{G}(t)\succ a\mathbb{S}_n$ if the weight of the edge k satisfies $\varepsilon_k(t)>ab'_k$ for $k=1,\ldots,m$. From Lemma 2 one has $\lambda_2(\mathbb{G}(t))>\lambda_2(a\mathbb{S}_n)=a\lambda_2(\mathbb{S}_n)=a$ if $\varepsilon_k(t)>ab'_k$ for all k. From Theorem 3, the synchronization manifold of the dynamical system (1) is globally asymptotically stable, if $\varepsilon_k(t)>ab'_k$ for $k=1,\ldots,m$. Thus we have proved the following theorem.

Theorem 5: Suppose that graph $\mathbb{G}(t)$ is undirected and connected. Under Assumption 1, the synchronization manifold of system (1) is globally asymptotically stable if the coupling strength of edge k satisfies $\varepsilon_k(t)>ab'_k$ for all $k=1,\ldots,m$ and for all t, where b'_k is given by (10).

Remark 3: In the above two cases for the edge $(1,i)$, one can choose arbitrarily the path in the topological graph $\mathbb{G}$ that connects vertices 1 and i. However, we prefer to choosing the shortest path(s). To specify the choice of these shortest paths, we set the rule as follows: for any two different vertices i and j in the topological graph $\mathbb{G}$, we consider the set of all the shortest paths connecting i and j, which is denoted by $\{\mathcal{P}_{ij}^{(1)},\mathcal{P}_{ij}^{(2)},\ldots,\mathcal{P}_{ij}^{(n_{ij})}\}$ with $n_{ij}\geq 1$. We choose a path in the set with equal probability $1/n_{ij}$. This rule is reasonable since the shortest paths are one of the most critical characterizations of connectivity between vertices in graphs and all the shortest paths between the same pair of vertices are usually equally important.

Remark 4: Compared with the graphical condition in Theorem 4, the method in Theorem 5 greatly reduces the computational complexity. There are only $n-1$ paths that need to be considered in our algorithm, while one needs to check $n(n-1)/2$ paths to apply Theorem 4.

Remark 5: Similar results can be obtained when $\mathbb{G}$ is compared with other graphs, such as rings and paths. Theorem 4 (respectively, Theorem 5) is the special case when graph $\mathbb{G}$ is compared with the complete graph $\mathbb{K}_n$ (respectively, the star graph $\mathbb{S}_n$). A proper choice of the graphs in comparison is helpful to obtain less conservative lower bounds for coupling strengths and reduce the computational complexity of the comparison at the same time.

Now we use one simple example to demonstrate the differences between the synchronization conditions in Theorems 5 and 4. Consider an undirected graph $\mathbb{G}_5$, consisting of five vertices $\{v_1,v_2,v_3,v_4,v_5\}$ and five edges $\{(1,2),(1,3),(1,4),(3,5),(4,5)\}$, which is shown on the left of Fig. 1. Suppose that $\mathbb{G}_5$ is compared with the star graph $\mathbb{S}_5$ on the right of Fig. 1. Comparing the two graphs, we use the edge (1,2) in $\mathbb{G}_5$ to represent the path connecting vertices v_1,v_2, (1,3) for v_1,v_3, (1,4) for v_1,v_4, and candidate paths (1,3,5) and (1,4,5) for vertices v_1,v_5. Then we have

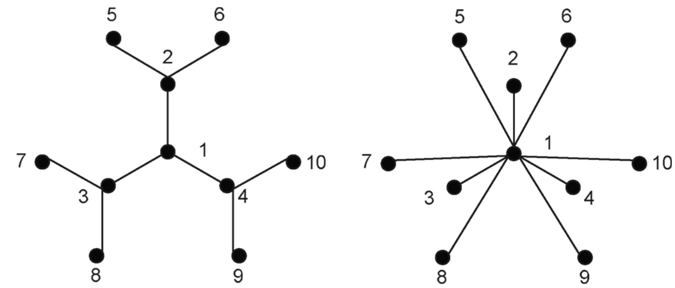

Fig. 2. Comparing a fractal tree with the star $\mathbb{S}_{10}$.

$$\begin{aligned}L_{\mathbb{S}_5}&=L_{(1,2)}+L_{(1,3)}+L_{(1,4)}+L_{(1,5)}\\&\preceq L_{(1,2)}+L_{(1,3)}+L_{(1,4)}\\&\quad+\left[\frac{1}{2}\left(2L_{(1,3)}+2L_{(3,5)}\right)+\frac{1}{2}\left(2L_{(1,4)}+2L_{(4,5)}\right)\right]\\&=L_{(1,2)}+2L_{(1,3)}+2L_{(1,4)}+L_{(3,5)}+L_{(4,5)}.\end{aligned}$$

Thus we have $b'_{(1,2)}=1$, $b'_{(1,3)}=2$, $b'_{(1,4)}=2$, $b'_{(3,5)}=1$, and $b'_{(4,5)}=1$. From Theorem 5, we obtain the bounds for the edges in $\mathbb{G}_5$: $\varepsilon_{(1,2)}\geq a$, $\varepsilon_{(1,3)}\geq 2a$, $\varepsilon_{(1,4)}\geq 2a$, $\varepsilon_{(3,5)}\geq a$, $\varepsilon_{(4,5)}\geq a$.

In comparison, now we compare the two graphs $\mathbb{G}_5$ and $\mathbb{K}_5$. We use edge (1,2) in $\mathbb{G}_5$ to represent the path connecting vertices v_1,v_2, edge (1,3) for vertices v_1,v_3, edge (1,4) for vertices v_1,v_4, edge (3,5) for vertices v_3,v_5, edge (4,5) for vertices v_4,v_5, and candidate paths (1,3,5) and (1,4,5) for vertices v_1,v_5, path (2,1,3) for vertices v_2,v_3, path (2,1,4) for vertices v_2,v_4, paths (2,1,4,5) and (2,1,3,5) for vertices v_2,v_5, paths (3,1,4) and (3,5,4) for vertices v_3,v_4. Then we have

$$\begin{aligned}L_{\mathbb{K}_5}&=L_{(1,2)}+L_{(1,3)}+L_{(1,4)}+L_{(1,5)}\\&\quad+L_{(2,3)}+L_{(2,4)}+L_{(2,5)}+L_{(3,4)}+L_{(3,5)}+L_{(4,5)}\\&\preceq L_{(1,2)}+L_{(1,3)}+L_{(1,4)}\\&\quad+\left[\frac{1}{2}\left(2L_{(1,4)}+2L_{(4,5)}\right)+\frac{1}{2}\left(2L_{(1,3)}+2L_{(3,5)}\right)\right]\\&\quad+\left(2L_{(1,2)}+2L_{(1,3)}\right)+\left(2L_{(1,2)}+2L_{(1,4)}\right)\\&\quad+\left[\frac{1}{2}\left(3L_{(1,2)}+3L_{(1,3)}+3L_{(3,5)}\right)\right.\\&\qquad\left.+\frac{1}{2}\left(3L_{(1,2)}+3L_{(1,4)}+3L_{(4,5)}\right)\right]\\&\quad+\left[\frac{1}{2}\left(2L_{(3,5)}+2L_{(4,5)}\right)+\frac{1}{2}\left(2L_{(1,3)}+2L_{(1,4)}\right)\right]\\&\quad+L_{(3,5)}+L_{(4,5)}\\&=8L_{(1,2)}+\frac{13}{2}L_{(1,3)}+\frac{13}{2}L_{(1,4)}+\frac{9}{2}L_{(3,5)}+\frac{9}{2}L_{(4,5)}.\end{aligned}$$

Thus we have $b_{(1,2)}=8$, $b_{(1,3)}=6.5$, $b_{(1,4)}=6.5$, $b_{(3,5)}=4.5$, and $b_{(4,5)}=4.5$. From Theorem 4, one of the possible sets of bounds is as follows: $\varepsilon_{(1,2)}\geq(8a/5)=1.6a$,

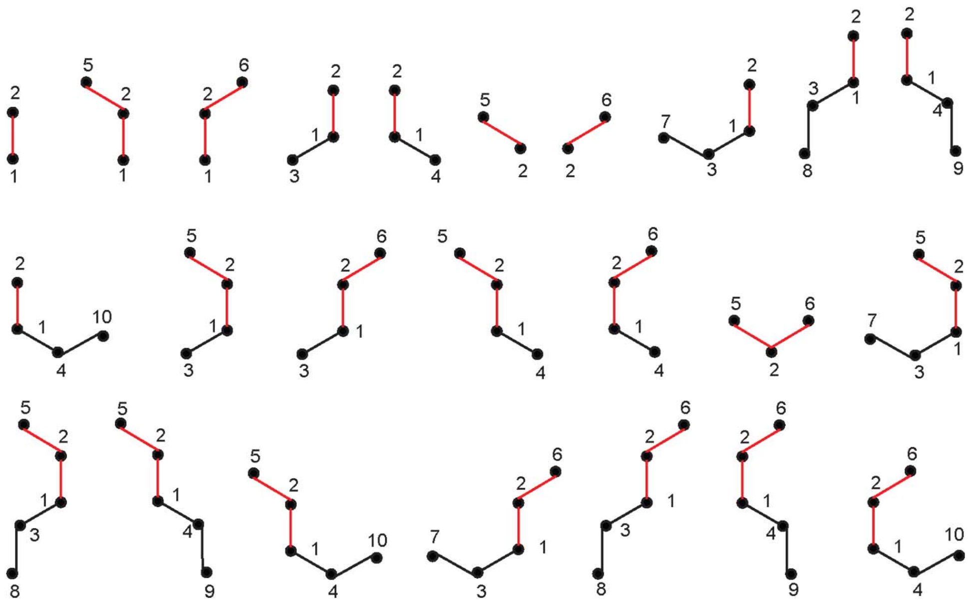

Fig. 3. Ppaths in $\mathbb{G}_{10}$ that pass through one of these edges (1,2), (2,5), (2,6).

$\varepsilon_{(1,3)} \geq (6.5a/5) = 1.3a$, $\varepsilon_{(1,4)} \geq (6.5a/5) = 1.3a$, $\varepsilon_{(3,5)} \geq (4.5a/5) = 0.9a$, and $\varepsilon_{(4,5)} \geq (4.5a/5) = 0.9a$.

From the above calculations, one can see that there are only 4 paths taken into consideration in $\mathbb{G}_5$ according to the proposed method in Theorem 5, while there are $(5 \times 4)/2 = 10$ paths considered according to the method in Theorem 4. In addition, we have obtained another set of bounds for the coupling strengths, in which $\varepsilon_{(1,2)}$ is smaller.

B. Benefits From Comparing With Stars

Now we give an example to demonstrate the advantages of using the synchronization condition in Theorem 5. To simplify our calculation, we consider a fractal tree with ten vertices, shown on the left of Fig. 2. First, let the fractal graph $\mathbb{G}_{10}$ be compared with the star graph $\mathbb{S}_{10}$. Because of the fractal structure of graph $\mathbb{G}_{10}$, we only need to focus on the calculations of bounds for the edges (1,2), (2,5), (2,6). And we have the comparison

$$\begin{aligned} L_{\mathbb{S}_{10}} &= L_{(1,2)} + L_{(1,5)} + L_{(1,6)} + \ldots \\ &\preceq L_{(1,2)} + 2(L_{(1,2)} + L_{(2,5)}) + 2(L_{(1,2)} + L_{(2,6)}) + \ldots \\ &= 5L_{(1,2)} + 2L_{(2,5)} + 2L_{(2,6)} + \ldots. \end{aligned}$$

Thus we have $b'_{(1,2)} = 5$, $b'_{(1,3)} = 2$, and $b'_{(2,6)} = 2$. From Theorem 5, we obtain the bounds for the edges (1,2), (2,5), (2,6) in graph $\mathbb{G}_{10}$

$$\varepsilon_{(1,2)} \geq 5a, \quad \varepsilon_{(2,5)} \geq 2a, \quad \varepsilon_{(2,6)} \geq 2a. \tag{11}$$

Second, we give another set of bounds for the edges in $\mathbb{G}_{10}$ using the method in Theorem 4. We implement graph comparison between graph $\mathbb{G}_{10}$ and the complete graph $\mathbb{K}_{10}$. Thus we need to consider the paths in $\mathbb{G}_{10}$ for every pair of vertices. The choice of the path in $\mathbb{G}_{10}$ for each pair of vertices is unique, because there is no cycle in $\mathbb{G}_{10}$. We only need to calculate the bounds for the edges (1,2), (2,5), (2,6). To do so, we first list all the possible paths that pass through at lease one of these edges, which are shown in Fig. 3. Then, from $b_k = \sum_{j>i; k\in\mathcal{P}_{ij}} |\mathcal{P}_{ij}|$ in Theorem 4, we have

$$\begin{aligned} b_{(1,2)} &= |\mathcal{P}_{1,2}| + |\mathcal{P}_{1,5}| + |\mathcal{P}_{1,6}| + |\mathcal{P}_{2,3}| + |\mathcal{P}_{2,4}| \\ &\quad + |\mathcal{P}_{2,7}| + |\mathcal{P}_{2,8}| + |\mathcal{P}_{2,9}| + |\mathcal{P}_{2,10}| \\ &\quad + |\mathcal{P}_{3,5}| + |\mathcal{P}_{3,6}| + |\mathcal{P}_{4,5}| + |\mathcal{P}_{4,6}| \\ &\quad + |\mathcal{P}_{5,7}| + |\mathcal{P}_{5,8}| + |\mathcal{P}_{5,9}| + |\mathcal{P}_{5,10}| \\ &\quad + |\mathcal{P}_{6,7}| + |\mathcal{P}_{6,8}| + |\mathcal{P}_{6,9}| + |\mathcal{P}_{6,10}| \\ &= 1 + 2\times 4 + 3\times 8 + 4\times 8 = 65. \end{aligned}$$

Following the same reasoning, we have

$$\begin{aligned} b_{(2,5)} &= |\mathcal{P}_{1,5}| + |\mathcal{P}_{2,5}| + |\mathcal{P}_{3,5}| + |\mathcal{P}_{4,5}| + |\mathcal{P}_{5,6}| \\ &\quad + |\mathcal{P}_{5,7}| + |\mathcal{P}_{5,8}| + |\mathcal{P}_{5,9}| + |\mathcal{P}_{5,10}| \\ &= 2 + 1 + 3 + 3 + 2 + 4\times 4 = 27 \end{aligned}$$

and $b_{(2,6)} = 27$ can be calculated similarly.

According to $\varepsilon_k > (b_k/n)a$ in Theorem 4, we obtain the bounds for the coupling strengths of the edges (1,2), (2,5), and (2,6) as

$$\begin{aligned} &\varepsilon_{(1,2)} \geq \frac{b_{(1,2)}}{10}a = 6.5a, \\ &\varepsilon_{(2,5)} \geq \frac{b_{(2,5)}}{10}a = 2.7a, \quad \varepsilon_{(2,6)} \geq \frac{b_{(2,6)}}{10}a = 2.7a. \end{aligned} \tag{12}$$

The above calculations show that the computational complexity of graph comparisons is greatly reduced by using the method in Theorem 5, comparing with what obtained using Theorem 4. In addition, we have obtained another set of bounds for coupling strengths of $\mathbb{G}_{10}$, in which each bound is much smaller. The proposed method is especially effective when networks are large and sparse.

Furthermore, the method we proposed can be applied to adaptively adjust the allocation of coupling strengths in order to ensure the synchronization of a dynamical network when its

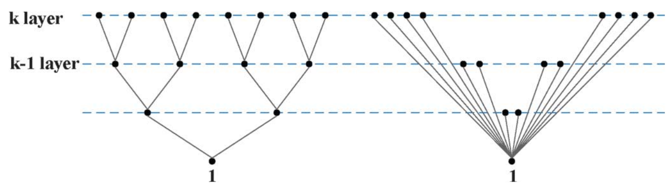

Fig. 4. Compare the growing complete binary tree with the growing star.

topology changes with time. We explore in this direction in the next subsection.

C. Applications in Networks With Continuing Growth

Now we apply Theorem 5 to allocate coupling strengths in networks with continuing growth. It is much easier to compare a growing graph with a growing star, with the same vertex set, than with a growing complete graph. If one more vertex is added to the star graph $\mathbb{S}_n$, only one more edge needs to be added. However, if one more vertex is added to the complete graph $\mathbb{K}_n$, n new edges need to get involved in calculation. Moreover, the second smallest eigenvalue of star graphs is the constant 1. These motivate us to apply Theorem 5 on dynamic networks with continuing growth.

In the following, we use complete binary trees to illustrate the application. The complete binary tree $\mathbb{T}_n$ with $n = 2^d - 1$ vertices is the graph with the edges of the form $(u, 2u)$ and $(u, 2u+1)$ for integer $u < n/2$ [18]. The complete binary tree $\mathbb{T}_n$ is shown on the left of Fig. 4.

Now we start with the complete binary tree with two layers $d = 2$. Comparing it with the star $\mathbb{S}_3$, we have

$$L_{\mathbb{S}_3} = L_{(1,2)} + L_{(1,3)}.$$

From Theorem 5, it is easy to obtain the bounds for the edges in $\mathbb{T}_3$: $\varepsilon_{(1,2)} \geq a$ and $\varepsilon_{(1,3)} \geq a$.

Compare the complete binary tree with $d = 3$ with $\mathbb{S}_7$

$$\begin{aligned} L_{\mathbb{S}_7} =& L_{(1,2)} + L_{(1,3)} + L_{(1,4)} + L_{(1,5)} + L_{(1,6)} + L_{(1,7)} \\ \preceq& L_{(1,2)} + L_{(1,3)} + 2[L_{(1,2)} + L_{(2,4)}] \\ &+ 2\left[L_{(1,2)} + L_{(2,5)}\right] + 2\left[L_{(1,3)} + L_{(3,6)}\right] \\ &+ 2\left[L_{(1,3)} + L_{(3,7)}\right] \\ =& (1 + 2 \times 2)L_{(1,2)} + (1 + 2 \times 2)L_{(1,3)} \\ &+ 2L_{(2,4)} + 2L_{(2,5)} + 2L_{(3,6)} + 2L_{(3,7)} \end{aligned}$$

and for the complete binary tree with $d = 4$

$$\begin{aligned} L_{\mathbb{S}_{15}} =& L_{(1,2)} + L_{(1,3)} + L_{(1,4)} + \ldots + L_{(1,15)} \\ \preceq& (1+2 \times 2+3 \times 2^2)L_{(1,2)} + (1+2 \times 2+3 \times 2^2)L_{(1,3)} \\ &+ (2 + 3 \times 2)L_{(2,4)} + (2 + 3 \times 2)L_{(2,5)} \\ &+ (2 + 3 \times 2)L_{(3,6)} + (2 + 3 \times 2)L_{(3,7)} \\ &+ 3L_{(4,8)} + 3L_{(4,9)} + 3L_{(5,10)} + 3L_{(5,11)} \\ &+ 3L_{(6,12)} + 3L_{(6,13)} + 3L_{(7,14)} + 3L_{(7,15)}. \end{aligned}$$

In this subsection we use $b'(k-1, k, d)$ to denote the value of b' for the edges between the $(k-1)$th layer and the kth layer in the binary tree $\mathbb{T}_n$ with $n = 2^d - 1$, and $\varepsilon(k-1, k, d)$ to denote the couplings between the $(k-1)$th layer and the kth layer in $\mathbb{T}_n$. We obtain the weights for the edges in the complete binary tree $\mathbb{T}_n$ with $n = 2^d - 1$ by induction, and we postulate

$$b'(k-1, k, d) = \sum_{j=k}^{d} (j-1) \times 2^{j-k} \tag{13}$$

for the edges of the form $(u, 2u)$ and $(u, 2u+1)$ with $u = 2^{k-2}, \ldots, 2^{k-1} - 1$ where $2 \leq k \leq d$ (the edges from the $(k-1)$th layer to the kth layer).

Now we use induction to prove our conjecture. Suppose that (13) holds for the complete binary tree $\mathbb{T}_n$ with $n = 2^d - 1$. Then we calculate $b'(k-1, k, d+1)$ for the binary tree with $n' = 2^{d+1} - 1$. The depth of $\mathbb{T}_{n'}$ is $d+1$. The binary tree $\mathbb{T}_{n'}$ has one more layer, and has 2^d more vertices which are labeled by $v_{2^d}, v_{2^d+1}, \ldots, v_{2^{d+1}-1}$. In order to assign couplings in $\mathbb{T}_{n'}$, we compare $\mathbb{T}_{n'}$ with the star graph $\mathbb{S}_{2^{d+1}-1}$, and calculate for the edges of the form $(u, 2u)$ and $(u, 2u+1)$ with $u = 2^{k-2}, \ldots, 2^{k-1} - 1$ and $2 \leq k \leq d+1$ (the edges from the $(k-1)$th layer to the kth layer):

$$\begin{aligned} b'(k-1, k, d+1) &= b'(k-1, k, d) + d \times 2^{d-k+1} \\ &= \sum_{j=k}^{d} (j-1) \times 2^{j-k} + d \times 2^{d-k+1} \\ &= \sum_{j=k}^{d+1} (j-1) \times 2^{j-k}. \end{aligned}$$

This shows that (13) still holds for the binary tree $\mathbb{T}_{n'}$ with $d+1$ layers, and hence by induction we have proved that (13) is correct. Thus, for the binary tree $\mathbb{T}_n$ with $n = 2^d - 1$, the weights for the edges of the form $(u, 2u)$ and $(u, 2u+1)$ for $u = 2^{k-2}, \ldots, 2^{k-1} - 1$ where $2 \leq k \leq d$ (the edges from the $(k-1)$th layer to the kth layer) should satisfy

$$\varepsilon(k-1, k, d) \geq \sum_{j=k}^{d} (j-1) \times 2^{j-k} a.$$

Now we state the result just proved as a proposition.

Proposition 1: Under Assumption 1, for the complete binary tree $\mathbb{T}_n$ with $n = 2^d - 1$ vertices, the global synchronization of system (1) is guaranteed if the couplings between the $(k-1)$th layer and the kth layer satisfy

$$\varepsilon(k-1, k, d) \geq \sum_{j=k}^{d} (j-1) \times 2^{j-k} a, \qquad 2 \leq k \leq d.$$

V. Applications to Synchronizability

In this section, we look at how to construct a lower bound for $\lambda_2(L_{\mathbb{G}})$ when the weights of $\mathbb{G}$ is fixed and given beforehand. In this case, $\lambda_2(L_{\mathbb{G}})$ is referred to as the algebraic connectivity [20] of $\mathbb{G}$ and describes how well $\mathbb{G}$ is connected; it has also been used to measure the synchronizability of a coupled dynamical network [10]. However, it is usually not so easy to calculate $\lambda_2(\mathbb{G})$ using local information and moreover, the existing algorithms can be computationally costly to implement [25]. In the following, we propose a way to construct a lower bound for $\lambda_2(\mathbb{G})$ using the pairwise path information of $\mathbb{G}$, which is inspired by the graph comparisons done in Theorems 3, 4, and 5.

A. Measure Synchronizability of Unweighted Graphs

We first assume that $\mathbb{G}$ is unweighted and time-invariant, and construct a lower bound for $\lambda_2(\mathbb{G})$ using b_k defined in Theorem 4.

Theorem 6: a) Let $b_{\max}$ denote $\max_{1\le k\le m} b_k$, where $b_k = \sum_{j>i;k\in\mathcal{P}_{ij}} |\mathcal{P}_{ij}|$. It holds that

$$\lambda_2(L_{\mathbb{G}}) \ge \frac{n}{b_{\max}}.$$

b) Let $b'_{\max}$ denote $\max_{1\le k\le m} b'_k$, where b'_k is defined by (10). It holds that

$$\lambda_2(L_{\mathbb{G}}) \ge \frac{1}{b'_{\max}}.$$

Proof:

a) We compare the complete graph $\mathbb{K}_n$ with the union of all the possible paths in $\mathbb{G}$. In the proof of Theorem 4, we have proven that

$$L_{\mathbb{K}_n} \preceq \sum_{k=1}^{m} b_k L_k, \quad \text{where } b_k = \sum_{j>i;k\in\mathcal{P}_{ij}} |\mathcal{P}_{ij}|.$$

Since $b_{\max} = \max_{1\le k\le m} b_k$, one has

$$L_{\mathbb{K}_n} \preceq b_{\max}\sum_{k=1}^{m} L_k = b_{\max} L_{\mathbb{G}}.$$

From Theorem 3, $b_{\max}\mathbb{G} \succeq \mathbb{K}_n$ is equivalent to $\lambda_2(L_{\mathbb{G}}) \ge (\lambda_2(L_{\mathbb{K}_n})/b_{\max}) = n/b_{\max}$.

b) We compare the star $\mathbb{S}_n$ with the union of the paths $(1,j), j=2,\ldots,n$ in $\mathbb{G}$. In the proof of Theorem 5, we have shown that

$$L_{\mathbb{S}_n} \preceq \sum_{k=1}^{m} b'_k L_k$$

where b'_k is defined by (10). Since $b'_{\max} = \max_{1\le k\le m} b'_k$, one has

$$L_{\mathbb{S}_n} \preceq b'_{\max}\sum_{k=1}^{m} L_k = b'_{\max} L_{\mathbb{G}}.$$

From Theorem 3, one has $\lambda_2(L_{\mathbb{G}}) \ge (\lambda_2(L_{\mathbb{S}_n})/b'_{\max}) = 1/b'_{\max}$. ■

Remark 6: The constructions of a lower bound of $\lambda_2(L_{\mathbb{G}})$ in [15] and [26] are similar to our estimation given by Theorem 6 a). However, we have taken a different approach, following a simpler derivation.

Remark 7: Theorem 6 b) is obtained through comparing graph $\mathbb{G}$ with the star $\mathbb{S}_n$. Other lower bounds can be obtained similarly if graph $\mathbb{G}$ is compared with graphs whose second smallest eigenvalues are known beforehand. A proper choice of the compared graphs is helpful to obtain tighter lower bounds for $\lambda_2(\mathbb{G})$ and reduce the computational complexity of comparison. In general, Theorem 6 b) is more efficient than a) especially when $\mathbb{G}$ is sparse and large.

Now we give an example to show the effectiveness of the estimations in Theorem 6. We consider the unweighted fractal graph whose topology is shown on the left of Fig. 2. From the calculations in Section IV-B, we have $b_{\max} = \max\{65, 27\} = 65$ and $b'_{\max} = \max\{5, 2\} = 5$. Then the calculated lower bounds for λ_2 are 0.1538 and 0.2 according to Theorem 6 a) and Theorem 6 b) respectively. The actual value of λ_2 of this graph is 0.2679. In comparison, one can obtain the lower bound $4/(10\times 4) = 0.1$ using Mohar's lower bound [27] $\lambda_2 \ge 4/nD_{\max}$ where $D_{\max}$ is the diameter of the graph.

B. Measure Synchronizability of Weighted Graphs

There have been different methods [28] to estimate the second smallest eigenvalues of the Laplacian matrices of unweighted graphs, but there is few result for weighted graphs. In this subsection, we measure the synchronizability of a weighted network by expanding the result in the previous subsection.

Theorem 7: Let the weights of the m edges of $\mathbb{G}$ be $c_1, c_2, \ldots, c_m$. Let

$$b^*_k \triangleq \sum_{j>i,k\in\mathcal{P}_{ij}} \left(\sum_{h\in\mathcal{P}_{ij}} \frac{1}{c_h}\right), \quad \text{for } 1\le k\le m. \tag{14}$$

It holds that $\lambda_2(L_{\mathbb{G}}) \ge n/b^*_{\max}$, where $b^*_{\max}$ is the maximum of all b^*_k.

Proof: For each pair of (i,j) where $j>i$, we choose one path in the weighted graph $\mathbb{G}$ with two associated vertices i, j. Then from Lemma 3, we have

$$L_{(i,j)} \preceq \left(\sum_{k\in\mathcal{P}_{ij}} \frac{1}{c_k}\right) \sum_{k\in\mathcal{P}_{ij}} c_k L_k.$$

We compare the complete graph $\mathbb{K}_n$ with the union of all possible paths in the weighted graph $\mathbb{G}$. Thus we have

$$\begin{aligned} L_{\mathbb{K}_n} &= \sum_{j>i} L_{(i,j)} \\ &\preceq \sum_{j>i}\left(\left(\sum_{k\in\mathcal{P}_{ij}} \frac{1}{c_k}\right) \sum_{k\in\mathcal{P}_{ij}} c_k L_k\right) \\ &= \sum_{k=1}^{m}\left(\sum_{j>i,k\in\mathcal{P}_{ij}}\left(\sum_{h\in\mathcal{P}_{ij}} \frac{1}{c_h}\right)\right) c_k L_k \\ &= \sum_{k=1}^{m} b^*_k c_k L_k \\ &\preceq b^*_{\max}\sum_{k=1}^{m} c_k L_k = b^*_{\max} L_{\mathbb{G}}. \end{aligned}$$

From Theorem 3, $b^*_{\max}\mathbb{G} \succeq \mathbb{K}_n$ is equivalent to

$$\lambda_2(L_{\mathbb{G}}) \ge \frac{\lambda_2(L_{\mathbb{K}_n})}{b^*_{\max}} = \frac{n}{b^*_{\max}}.$$

Thus we have arrived at the conclusion. ■

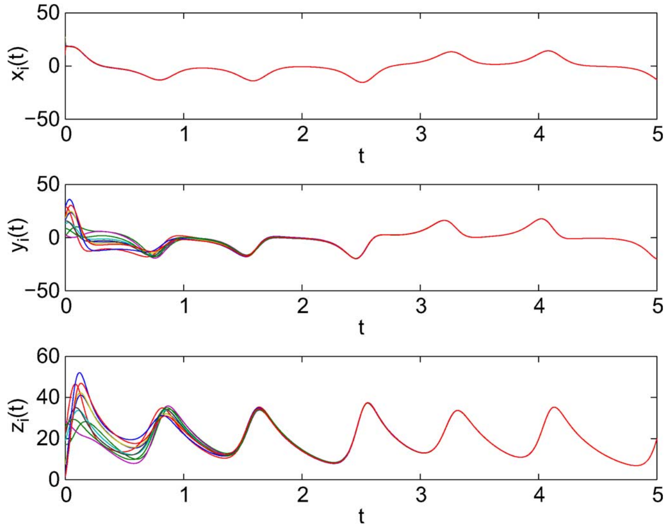

Fig. 5. States of the coupled Lorenz oscillators (15).

Remark 8: Theorem 6 a) is a special case of Theorem 7, which can be verified by setting $c_1 = c_2 = \ldots = c_m = 1$ in (14). In this case one obtains $b_k^* = \sum_{j>i,k\in\mathcal{P}_{ij}} |\mathcal{P}_{ij}|$ from (14). In addition, Theorem 7 is obtained through comparing the weighted graph $\mathbb{G}$ with the complete graph with the same vertex set. Other lower bounds can be obtained similarly if $\mathbb{G}$ is compared with other fundamental graphs whose second smallest eigenvalues are known beforehand.

Now we give an example to show how to use graph comparison to estimate lower bounds for synchronizability of weighted graphs. We again consider the fractal graph $\mathbb{G}_{10}$ whose topology is shown on the left of Fig. 2, however all the edges in the graph are weighted in this example. Suppose the weights of the edges (1,2), (1,3), (1,4) are c_1, and the weights of the other edges are c_2. To simplify the calculations in graph comparison, we choose to compare the fractal graph $\mathbb{G}_{10}$ with the star $\mathbb{S}_{10}$, rather than with the complete graph $\mathbb{K}_{10}$. In view of Lemma 3, we can compare edge (1,5) in $\mathbb{S}_{10}$ with the weighted edges (1,2), (2,5) in $\mathbb{G}_{10}$, and obtain that $L_{(1,5)} \preceq ((1/c_1)+(1/c_2))(c_1L_{(1,2)}+c_2L_{(2,5)})$. Similarly, we have $L_{(1,6)} \preceq ((1/c_1)+(1/c_2))(c_1L_{(1,2)}+c_2L_{(2,6)})$.

Because of the fractal structure of $\mathbb{G}_{10}$, we only need to focus on the calculations for the edges (1,2), (2,5), (2,6) and thus obtain

$$
\begin{aligned}
L_{\mathbb{S}_{10}} &= L_{(1,2)} + L_{(1,5)} + L_{(1,6)} + \ldots \\
&\preceq L_{(1,2)} + \left(\frac{1}{c_1}+\frac{1}{c_2}\right)\left(c_1L_{(1,2)}+c_2L_{(2,5)}\right) \\
&\quad + \left(\frac{1}{c_1}+\frac{1}{c_2}\right)\left(c_1L_{(1,2)}+c_2L_{(2,6)}\right) + \ldots \\
&= \left(\frac{3}{c_1}+\frac{2}{c_2}\right)c_1L_{(1,2)} + \left(\frac{1}{c_1}+\frac{1}{c_2}\right)c_2L_{(2,5)} \\
&\quad + \left(\frac{1}{c_1}+\frac{1}{c_2}\right)c_2L_{(2,6)} + \ldots .
\end{aligned}
$$

So we have $b'_{\max} = \max\{(3/c_1)+(2/c_2),(1/c_1)+(1/c_2)\} = (3/c_1)+(2/c_2)$. Note that the Laplacian matrix of $\mathbb{G}_{10}$ can be written as $L_{\mathbb{G}_{10}} = c_1[L_{(1,2)}+L_{(1,3)}+L_{(1,4)}] + c_2[L_{(2,5)} + L_{(2,6)}+L_{(3,7)} + L_{(3,8)} + L_{(4,9)} + L_{(4,10)}]$. Then one has $L_{\mathbb{S}_{10}} \preceq b'_{\max}L_{\mathbb{G}_{10}}$. Therefore $\lambda_2(L_{\mathbb{G}_{10}}) \geq (\lambda_2(\mathbb{S}_{10})/b'_{\max}) = c_1c_2/(2c_1+3c_2)$.

VI. Numerical Simulation

In this section, we provide a numerical example to validate Theorem 5. Given the self-dynamics $f(\cdot)$ and the inner coupling matrix P, we first need to figure out the value a in Assumption 1. This has been extensively studied in the literature on control and synchronization of chaotic dynamical systems [29], [30]. For instance, two mutually coupled Chua's circuits can synchronize by choosing $P = \operatorname{diag}\{1,0,0\}$ for a large enough scalar $a > \max(-G_a,-G_b)/C_1$ [29, Cor. 10], where G_a, G_b, C_1 are parameters of Chua's circuits. As another example, two Lorenz systems mutually coupled through the first component of their states can synchronize when a is greater than a computable threshold [14, App. A]. In this simulation, we consider the network (1) consisting of n Lorenz systems coupled through the first components of their states. To be specific, the dynamics of the network are given by

$$
\begin{cases}
\dot{x}_i = \sigma(y_i - x_i) + \sum_{j=1}^{n} \varepsilon_{ij}(t)x_j \\
\dot{y}_i = rx_i - y_i - x_iz_i \\
\dot{z}_i = -bz_i + x_iy_i
\end{cases}
\tag{15}
$$

and the inner coupling matrix is $P = \operatorname{diag}\{1,0,0\}$. According to [14, App. A], the quantity $a > a^* = (b(b+1)(r+\sigma)^2/16(b-1)) - \sigma$. We choose the fractal graph with $n = 10$ vertices on the left of Fig. 2 to be the network topology used in the simulation. The bounds for coupling strengths have been calculated and given by (11), and so we set the coupling strengths $\varepsilon_{(1,2)} = \varepsilon_{(1,3)} = \varepsilon_{(1,4)} = 5a$, and the coupling strengths for the other edges $2a$. The parameters in (15) are set to be $\sigma = 10$, $r = 25$, $b = 8/3$. The initial states are randomly chosen from [0,30]. The three subfigures in Fig. 5 show the state of the coupled network (15) in its x, y, z-dimension respectively. From Fig. 5, one can see that the coupled Lorenz oscillators asymptotically synchronize by adopting the coupling strength allocation (11) obtained according to Theorem 5. The simulation results illustrate the correctness of the theoretical analysis in Section IV-B.

VII. Conclusions

In this paper we have presented new ways to allocate coupling strengths using spectral graph theory in order to achieve synchronization in complex networks. The main idea is to bound the second-smallest eigenvalues of the Laplacian matrices associated with the given networks by comparing the corresponding network graphs to complete or other typical graphs with the same vertex sets. The obtained results can simplify the computation and be applied to growing networks.

Currently, we are looking into applying the proposed methodologies to networks with directed topologies, some preliminary results have been presented in [31]. The main challenge is then how to deal with the fact that the Laplacian matrices associated with directed graphs are not guaranteed to be positive semidefinite anymore. We are also working on using the constructed synchronization criteria to develop optimal or suboptimal solutions for adding or deleting edges in a network to achieve better synchronizability. It is of great interest to apply our results to practical engineered complex networks, such as the synchronization of generators in electric power grids and data fusion for signal processing in sensor networks.

APPENDIX

In this appendix, we show that Assumption 1 is equivalent to Belykh *et al.*'s assumption on self-dynamics $f(\cdot)$ of oscillators in [14]. The assumption in [14] requires

$$(x_j - x_i)^T \left[\int_0^1 Df\left(\beta x_j + (1-\beta)x_i\right) d\beta - aP \right] (x_j - x_i) < 0, \quad \text{for any } x_i \neq x_j \tag{16}$$

where $f(\cdot)$ is continuously differentiable.

Note that $(d/d\beta)f(\beta x_j + (1-\beta)x_i) = [Df(\beta x_j + (1-\beta)x_i)](x_j - x_i)$. The following equation holds [14]:

$$\begin{aligned} f(x_j) - f(x_i) &= \int_0^1 \frac{d}{d\beta} f\left(\beta x_j + (1-\beta)x_i\right) d\beta \\ &= \left[\int_0^1 Df\left(\beta x_j + (1-\beta)x_i\right) d\beta \right] (x_j - x_i). \end{aligned}$$

Then we have

$$\begin{aligned} &(x_j - x_i)^T \left[\int_0^1 Df\left(\beta x_j + (1-\beta)x_i\right) d\beta - aP \right] (x_j - x_i) \\ &= (x_j - x_i)^T \left[(f(x_j) - f(x_i)) - aP(x_j - x_i) \right]. \end{aligned}$$

Therefore, $(x_j - x_i)^T[(f(x_j) - f(x_i)) - aP(x_j - x_i)] < 0$ in Assumption 1 implies the inequality (16), provided that $f(\cdot)$ is continuously differentiable.

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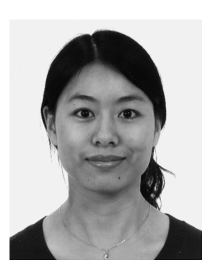

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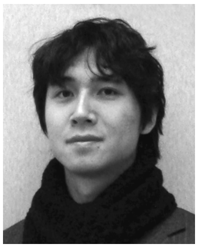

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