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Forming Circle Formations of Anonymous Mobile Agents With Order Preservation

Chen Wang, *Student Member, IEEE*, Guangming Xie, *Member, IEEE*, and Ming Cao, *Member, IEEE*

***Abstract*—We propose distributed control laws for a group of anonymous mobile agents to form desired circle formations when the agents move in the one-dimensional space of a circle. The agents are modeled by kinematic points. They share the common knowledge of the orientation of the circle, but are oblivious and anonymous. Moreover, each agent can only sense the relative positions of its neighboring two agents that are immediately in front of or behind itself. Distributed control strategies are designed for the agents using only the information of the relative positions of their two neighbors and also the given desired distances to its neighboring two agents. To make the control strategies more practical, we discuss the corresponding sampled-data control laws, and utilizing the technique of adopting time-varying gains, we obtain control laws that are able to guide the agents to form the desired circle formation within any given finite time. One feature of the proposed control laws is that they guarantee that the spatial ordering of the agents are preserved throughout the system's evolution, and thus no collision may take place during the process of forming circle formations. Both theoretical analysis and numerical simulations are given to show the effectiveness of the proposed formation control strategies.**

***Index Terms*—Circle formation, distributed control, finite-time convergence, multi-agent system, order preservation, sampled-data control.**

I. Introduction

Cooperative mobile robots have been utilized more and more often to carry out a growing variety of team tasks, such as environmental monitoring [2], surveillance [3], exploration [4], pursuit and evasion [5], search and rescue [6], and transportation [7]. One active research topic that arises in such robotic applications is the pattern-forming problem, where autonomous mobile agents are required to generate and maintain cooperatively desired geometric patterns that are useful for various team tasks [8], [9]. In this line of research, significant efforts have been made on the development of distributed strategies guiding agents to form circle formations [10]; in particular, the focus is how to lead the agents to distribute evenly on a given circle.

In theoretical computer science, the so called *semi-synchronous* model developed in [10] has become popular and motivated quite a number of following works [11]–[13]. It has been proposed that the circle-forming problem can be decomposed into two independent subproblems: one is to guide the agents to move on a circle and the other to arrange them in positions evenly distributed on the circle. Among these works, it is usually assumed that the agents are i) oblivious, namely without memories about past actions and observations, ii) anonymous, namely not distinguishable from one another, iii) unable to communicate directly, and iv) can only interact through sensing other agents' positions. Later on, the circle-forming problem has been further studied in [14] in a complete asynchronous setting but requiring that all the agents can only move on a circle.

Research efforts have also been made in the systems and control community on the circle-forming problem [8]. For example, Marshall *et al.* have studied distributed control laws under which agents generate circular pursuit patterns [15]. There are still open questions that are motivated by the implementation of such control laws. For example, people want to know whether desired formations can be obtained in finite time instead of asymptotically; similar finite-time convergence questions have been addressed for consensus-type algorithms [16]–[18]. We have recently considered the scenario when agents are under locomotion constraints [19].

The goal of this paper is to design distributed control laws that can guide a group of autonomous mobile agents that move on a circle to form any given circle formations. The spatial ordering of the agents need to be preserved to avoid collisions between agents, which makes the strategies more attractive when they are implemented in real robots. To be more specific, we consider a system consisting of multiple mobile agents modeled by point masses, all of which move in the one-dimensional space of a given circle. The agents are oblivious, anonymous, and unable to communicate directly; they share the common notion of being clockwise on the circle. Each agent can only sense the relative angular positions of its neighboring two agents that are immediately in front of or behind itself. Then the graph describing the neighbor relationships between the agents is always a ring [20]. After studying the performances of the control law that we propose to solve the formulated circle-forming problem, we further investigate its variation in the form of a sampled-data control law to meet needs from practice. In the end, motivated by our recent work [21] on finite-time convergence of consensus algorithms through linear time-varying feedback, we look into control laws that can guarantee that the agents form prescribed circle formations within any given finite time.

The main contribution of the paper is threefold. First, we study the circle-forming problem without the requirement that all the desired distances between neighboring agents are equal. Second, we take into account two requirements from real robotic applications about using sampled data and generating a formation within finite time. Third, we have identified and studied the order preservation property that is particularly useful to prevent collisions between agents. The paper is organized as follows. In Section II, we formulate the circle formation problem. Then we propose a distributed control law and analyze its performances in Section III. In Section IV, a sampled-data control law

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C. Wang is with the Center for Systems and Control, State Key Laboratory for Turbulence and Complex Systems, College of Engineering, Peking University, Beijing 100871, China and also with the Faculty of Mathematics and Natural Sciences, University of Groningen, Groningen 9747 AG, The Netherlands (e-mail address: wangchen@pku.edu.cn).

G. Xie is with the Center for Systems and Control, State Key Laboratory for Turbulence and Complex Systems, College of Engineering, Peking University, Beijing 100871, China (e-mail: xiegming@pku.edu.cn; xieguangming@tsinghua.org.cn).

M. Cao is with the Faculty of Mathematics and Natural Sciences, University of Groningen, Groningen 9747 AG, The Netherlands (e-mail: ming.cao@ieee.org).

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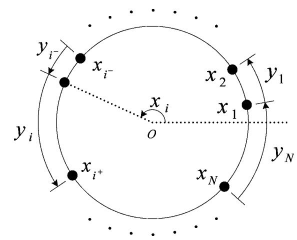

Fig. 1. Agents distributed on a circle.

is studied and in Section V, control laws guaranteeing finite-time convergence are designed for systems without or with sampled data. Simulations results are given in Section VI.

II. Problem Formulation

We consider a group of N, $N \geq 2$, agents that move on a given circle. The agents are initially positioned on the circle already and no two agents occupy the same position. The agents share the common notion of being clockwise and for analysis purposes we label the agents counterclockwise, as shown in Fig. 1, by $1, 2, \ldots, N$. Also for analysis purposes, we denote the positions of agent i, $1 \leq i \leq N$, measured by angles in a preselected coordinate system by x_i and without loss of generality assume that the agents' initial positions satisfy

$$0 \leq x_1(0) < \cdots < x_i(0) < x_{i+1}(0) \cdots < x_N(0) < 2\pi. \quad (1)$$

Each agent can only sense the relative positions of its immediately neighboring two agents. Then the graph describing the neighbor relationships is an undirected ring $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E} = \{(1,2), (2,3), \ldots, (N-1, N), (N, 1)\}$. We denote agent i's two neighbors by i^+ and i^- following the rule:

$$i^+ = \begin{cases} i+1 & \text{when } i = 1, 2, \ldots, N-1 \\ 1 & \text{when } i = N \end{cases}$$

and

$$i^- = \begin{cases} N & \text{when } i = 1 \\ i-1 & \text{when } i = 2, 3, \ldots, N \end{cases}. \quad (2)$$

Each agent is described by a kinematic point

$$\dot{x}_i(t) = u_i(t), \qquad i = 1, 2, \ldots, N \quad (3)$$

where u_i is the control input. Let d_i denote the prescribed angular distance between agents i and i^+. Then the desired circle formation is determined completely by the vector

$$d = [d_1, d_2, \ldots, d_N]^T. \quad (4)$$

We say a desired circle formation is *admissible* if $d_i > 0$ and $\sum_{i=1}^N d_i = 2\pi$. We further introduce the variable y_i that describes the angular distance from agent i to its immediate counterclockwise neighboring agent. It can be obtained through local measurements, such as the reading of sensors installed on agent i. Since y_i is defined with respect to agent i's local coordinate system, one can assume that it always takes value from $[0, 2\pi)$. Consequently, at time $t = 0$, it holds that

$$y_i = \begin{cases} x_{i^+} - x_i & \text{when } i = 1, 2, \ldots, N-1 \\ x_{i^+} - x_i + 2\pi & \text{when } i = N \end{cases}. \quad (5)$$

Moreover, $\sum_{i=1}^N y_i(t) \equiv 2\pi$ always holds.

Now we are ready to formulate the circle formation problem.

Definition 1 (Circle Formation Problem): Given an admissible circle formation characterized by d, design distributed control laws $u_i(t) = u_i(y_i, y_{i^-}, d_i, d_{i^-})$, $i = 1, \ldots, N$, such that under any initial condition (1) the solution to system (3) converges to some equilibrium point x^* (dependent on $x(0)$) satisfying $y^* = d$. Moreover, the Circle Formation Problem becomes a Uniform Circle Formation Problem when $d = (2\pi/N)\mathbf{1}_N$ where $\mathbf{1}_N$ is the N-dimensional all-one vector.

In robotic applications, it is usually desirable that a robotic team task can be finished within finite time. This motivates us to formulate the finite-time circle formation problem.

Definition 2 (Finite-Time Circle Formation Problem): Given any finite time $t_f \in (0, +\infty)$, design distributed control laws $u_i(t) = u_i(t, y_i, y_{i^-}, d_i, d_{i^-})$, $t \in [0, t_f)$, $i = 1, \ldots, N$, such that the Circle Formation Problem is solved as $t \rightarrow t_f$.

Note that throughout the paper, we take the notation $\rightarrow$ to mean approaching from below.

Since the agents have been ordered counterclockwise on the circle, if the agents' ordering can be preserved throughout the system's evolution, then no collision may take place between agents. We define what we mean by preserving orders as follows.

Definition 3 (Order Preservation): For the N-agent system under consideration, we say the agents' spatial ordering is preserved under control laws $u_i(t)$ if with initial condition (1), the solution to system (3) satisfies $y(t) > 0$ throughout the system's evolution.

In the next section, we discuss our circle-forming control laws using the notion of way points.

III. Way-Point Control Law

In order to solve the Circle Formation Problem, it is natural to consider the strategy to let each agent i move towards its *way-point* that is determined completely by its two neighbors' relative positions and the prescribed distances d_i and d_{i^-}

$$\bar{x}_i(t) = x_{i^-}(t) + \frac{y_i(t) + y_{i^-}(t)}{d_i + d_{i^-}} d_{i^-}.$$

Obviously, if indeed $x_i(t) = \bar{x}_i(t)$, it must be true that the ratio of $|y_i(t)|$ over $|y_{i^-}(t)|$ is exactly d_i/d_{i^-}. Then the way-point based control law for agent i becomes

$$u_i(t) = \bar{x}_i(t) - x_i(t)$$

which can be further written into

$$u_i(t) = \frac{d_{i^-}}{d_i + d_{i^-}} y_i(t) - \frac{d_i}{d_i + d_{i^-}} y_{i^-}(t) \qquad i = 1, 2, \ldots, N. \quad (6)$$

Substituting (6) into (3), we arrive at the the resulting closed-loop dynamics of the N-agent system

$$\begin{cases} \dot{x}_1 = -x_1 + \frac{d_1}{d_1 + d_N} x_N + \frac{d_N}{d_1 + d_N} x_2 - \frac{2\pi d_1}{d_1 + d_N} \\ \dot{x}_i = -x_i + \frac{d_i}{d_i + d_{i-1}} x_{i-1} + \frac{d_{i-1}}{d_i + d_{i-1}} x_{i+1} \quad i = 2, \ldots, N-1, \\ \dot{x}_N = -x_N + \frac{d_N}{d_N + d_{N-1}} x_{N-1} + \frac{d_{N-1}}{d_N + d_{N-1}} x_1 + \frac{2\pi d_{N-1}}{d_N + d_{N-1}} \end{cases} \quad (7)$$

which can be rewritten equivalently using y_i's as

$$\dot{y}_i = \left(-\frac{d_{i^+}}{d_{i^+} + d_i} - \frac{d_{i^-}}{d_i + d_{i^-}} \right) y_i + \frac{d_i}{d_{i^+} + d_i} y_{i^+} + \frac{d_i}{d_i + d_{i^-}} y_{i^-}, \qquad i = 1, 2, \ldots, N. \quad (8)$$

$$L(d)=\begin{bmatrix} \frac{d_2}{d_2+d_1}+\frac{d_N}{d_1+d_N} & -\frac{d_1}{d_2+d_1} & 0 & \cdots & 0 & -\frac{d_1}{d_1+d_N} \\ -\frac{d_2}{d_2+d_1} & \frac{d_3}{d_3+d_2}+\frac{d_1}{d_2+d_1} & -\frac{d_2}{d_3+d_2} & \cdots & 0 & 0 \\ 0 & -\frac{d_3}{d_3+d_2} & \frac{d_4}{d_4+d_3}+\frac{d_2}{d_3+d_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{d_N}{d_N+d_{N-1}}+\frac{d_{N-2}}{d_{N-1}+d_{N-2}} & -\frac{d_{N-1}}{d_N+d_{N-1}} \\ -\frac{d_N}{d_1+d_N} & 0 & 0 & \cdots & -\frac{d_N}{d_N+d_{N-1}} & \frac{d_1}{d_1+d_N}+\frac{d_{N-1}}{d_N+d_{N-1}} \end{bmatrix} \tag{10}$$

We summarize the system dynamics into a compact form

$$\dot{y}(t)=-L(d)y(t) \tag{9}$$

where $y(t)=[y_1(t),y_2(t),\ldots,y_N(t)]^T$ and $L(d)$ is given by (10), as shown at the top of the page.

Now we go ahead analyzing the convergence of the closed-loop system (9). Towards this end, we first list some useful matrix analysis results. For a positive integer n, we use $\mathcal{M}_n$ to denote the set of all n-by-n real matrices. We say a matrix A is *nonnegative* (resp. *positive*), denoted by $A\geq 0$ (resp. $A>0$), if all its entries are nonnegative (resp. positive). The directed graph of a matrix $A\in\mathcal{M}_n$, denoted by $\mathbb{G}(A)$, is the directed graph with the vertex set $\{v_i\}, i\in\{1,2,\ldots,n\}$, such that there is a directed edge in $\mathbb{G}(A)$ from v_j to v_i if and only if $a_{ij}\neq 0$ [22]. A directed graph is said to be *strongly connected* if there is a directed path between any pair of distinct vertices [20].

Lemma 1 (Theorem 6.2.24 of [22]): For $A\in\mathcal{M}_n$, the following are equivalent:

i) A is irreducible;
ii) $\mathbb{G}(A)$ is strongly connected.

A nonnegative matrix $A\in\mathcal{M}_n$ is said to be *primitive* if it is irreducible and has only one eigenvalue of maximum modulus [22]. Then we have the following results.

Lemma 2 (Theorem 8.5.2 of [22]): If $A\in\mathcal{M}_n$ is nonnegative, then A is primitive if and only if $A^m>0$ for some integer $m\geq 1$.

Lemma 3 (Lemma 8.5.5 of [22]): If $A\in\mathcal{M}_n$ is nonnegative and irreducible, and if all the main diagonal entries of A are positive, then $A^{n-1}>0$.

Lemma 4 (Lemma 8.5.6 of [22]): Let $A\in\mathcal{M}_n$ be nonnegative and primitive. Then A^k is nonnegative, irreducible, and primitive for all $k=1,2,\ldots$.

Now we analyze the eigenvalues of $L(d)$, denoted by $\lambda_1,\ldots,\lambda_N$, that will be used later on.

Lemma 5: It holds that

i) $L(d)$ is diagonalizable and $\lambda_i\in[0,2], i=1,2,\ldots,N$;
ii) 0 is a single eigenvalue;
iii) When N is even, 2 is an eigenvalue, while when N is odd, 2 is not.

Proof: Let $\sqrt{D}$ denote the diagonal matrix $\mathrm{diag}\{\sqrt{d_1},\sqrt{d_2},\ldots,\sqrt{d_N}\}$. Then one can check that the matrix $\sqrt{D}^{-1}L(d)\sqrt{D}$ is a symmetric real matrix. Thus, $L(d)$ is diagonalizable and all its eigenvalues are real. One can further check that all the λ_i's are located within the union

$$\mathcal{G}(L)=\bigcup_{i=1}^{N}\{z\in\mathbb{R}:|z-l_{ii}|\leq R_i(L)\} \tag{11}$$

where

$$R_i(L)=\frac{d_i}{(d_{i+}+d_i)}+\frac{d_i}{(d_i+d_{i-})}=2-l_{ii}, \qquad 1\leq i\leq N. \tag{12}$$

And all the λ_i's are also located within the union

$$\mathcal{G}(L^T)=\bigcup_{i=1}^{N}\{z\in\mathbb{R}:|z-l_{ii}|\leq R_i(L^T)\} \tag{13}$$

where

$$R_i(L^T)=\frac{d_{i+}}{(d_{i+}+d_i)}+\frac{d_{i-}}{(d_i+d_{i-})}=l_{ii}, \qquad 1\leq i\leq N. \tag{14}$$

Since $0<l_{ii}<2$, it must be true that $\mathcal{G}(L)\subseteq\{z\in\mathbb{R}:|z|\leq 2\}$ and $\mathcal{G}(L^T)\subseteq\{z\in\mathbb{R}:|z-2|\leq 2\}$. Thus, $\mathcal{G}(L)\cap\mathcal{G}(L^T)\subseteq\{z\in\mathbb{R}:0\leq z\leq 2\}$. It follows then that $\lambda_i\in[0,2], i=1,2,\ldots,N$.

Consider the matrix $Q(d)=2I_N-L(d)$. Its eigenvalues are $2-\lambda_1,\ldots,2-\lambda_N$. Since $\lambda_i\in[0,2]$, we have the spectral radius $\rho(Q(d))=2$, and one can check that 2 is an eigenvalue of $Q(d)$. Since $\mathbb{G}(Q(d))$ is strongly connected, from Lemma 1, we know that $Q(d)$ is irreducible. Furthermore, $Q(d)\geq 0$ and all the main diagonal entries of $Q(d)$ are positive. Then from Lemma 3 and Lemma 2, $Q(d)$ is primitive, which implies that $Q(d)$ has only one eigenvalue of maximum modulus. Said differently, the largest eigenvalue of $Q(d)$ is single. Thus, in view of the relationship between $L(d)$ and $Q(d)$, we know that 0 is a single eigenvalue of $L(d)$.

Moreover, one can check that $\sqrt{D}^{-1}Q(d)\sqrt{D}$ is also a symmetric real matrix. For any $z=[z_1,z_2,\ldots,z_N]^T\in\mathbb{R}^N$, we have

$$z^T\sqrt{D}^{-1}Q(d)\sqrt{D}z=\sum_{i=1}^{N}\frac{(\sqrt{d_i}z_i+\sqrt{d_{i+}}z_{i+})^2}{(d_i+d_{i+})}\geq 0. \tag{15}$$

One can then further check that there must exist a nonzero $z\in\mathbb{R}^N$ such that $z^T\sqrt{D}^{-1}Q(d)\sqrt{D}z=0$ for even N, but there does not exist such a z for odd N. It follows that 0 is an eigenvalue of $Q(d)$ for even N and is not for odd N. Correspondingly, 2 is an eigenvalue of $L(d)$ for even N and is not for odd N. ■

In view of Lemma 5, without loss of generality, we now assume $\lambda_1=0<\lambda_2\leq\cdots\leq\lambda_N$ throughout the rest of this paper.

Now we prove the main result in this section.

Theorem 1: Given any admissible circle formation characterized by d, the Circle Formation Problem is solved with order preservation under the proposed control law (6).

Proof: Let $D=\mathrm{diag}\{d_1,d_2,\ldots,d_N\}$, and then $D^{-1}L(d)D=L^T(d)$. Let $\delta=D^{-1}y$, and then we have

$$\dot{\delta}(t)=-L^T(d)\delta(t). \tag{16}$$

Since $L^T(d)$ is the Laplacian matrix of $\mathbb{G}(L^T(d))$ which is strongly connected, we have

$$\lim_{t\to\infty}\delta(t)=c\mathbf{1}_N \tag{17}$$

where c is a constant. It follows then from the definition of δ that

$$\lim_{t\to\infty} y(t) = cd. \tag{18}$$

Noticing $\sum_{i=1}^{N} y_i(t) = 2\pi$ and $\sum_{i=1}^{N} d_i = 2\pi$, it must be true that $c = 1$ and thus

$$\lim_{t\to\infty} y(t) = d. \tag{19}$$

In other words, the Circle Formation Problem is solved under the proposed control law (6).

Furthermore, the solution to system (9) is

$$y(t) = e^{-L(d)t}y(0), \qquad t \geq 0. \tag{20}$$

Consider

$$e^{-L(d)t} = e^{-2I_N}e^{2I_N - L(d)t} = e^{-2}e^{Q(d)t} = e^{-2}\sum_{k=0}^{\infty}\frac{1}{k!}Q^k(d)t^k \tag{21}$$

where $Q(d) = 2I_N - L(d)$. Since $Q(d)$ is nonnegative and primitive, from Lemma 4, $Q^k(d)$ is nonnegative for all $k = 1, 2, \ldots$ Furthermore, from Lemma 2, $\sum_{k=0}^{\infty}(1/k!)Q^k(d)t^k$ is positive. Thus, $e^{-L(d)t}$ is positive for $t \geq 0$. The initial condition (1) ensures that $y(0) > 0$ because of the construction of y_i. So under the initial condition, any solution to system (9) satisfies $y(t) > 0$ for all $t \geq 0$. So the Circle Formation Problem is solved with order preservation. ■

The following result is a special case of Theorem 1.

Corollary 1: The Uniform Circle Formation Problem is solved with order preservation under the proposed control law that simplifies to

$$u_i(t) = \frac{y_i(t) - y_{i^-}(t)}{2} \qquad i = 1, 2, \ldots, N. \tag{22}$$

In the next sections, we consider practical issues arising when implementing the proposed control laws.

IV. Sampled-Data Based Way-Point Control Law

In the previous section, a distributed control law (6) has been proposed, which has been proven to solve the Circle Formation Problem with order preservation. In real multi-robot systems, continuous-time control laws may not be able to be implemented directly because of hardware constraints related to communication bandwidth, rise time, and computation load. Hence, sampled-data based control laws become more practical in such cases [23]. In this section, we investigate the convergence of the way-point control laws discussed earlier when sampled data are used. The sampled-data based control laws are developed using techniques of periodic sampling and zero-order hold.

Let $h > 0$ be the sampling period, then we propose to use the following sampled-data way-point control laws

$$u_i(t) = \frac{d_{i^-}}{d_i + d_{i^-}}y_i(kh) - \frac{d_i}{d_i + d_{i^-}}y_{i^-}(kh)$$
$$t \in [kh, kh+h),\ k = 0, 1, 2, \ldots;\ i = 1, 2, \ldots, N. \tag{23}$$

With the control law (23), the overall closed-loop system can be described by

$$y(kh + h) = P(d)y(kh), \quad k = 0, 1, 2, \ldots \tag{24}$$

where

$$P(d) = I_N - hL(d). \tag{25}$$

Necessary and sufficient conditions for the convergence of the overall system are as follows.

Theorem 2: Consider the sampled-data control law (23), the Circle Formation Problem is solved for all admissible circle formations characterized by d if and only if $0 < h < 1$ for even N and $0 < h \leq 1$ for odd N. Furthermore, the corresponding closed-loop system has the property of order preservation if and only if $0 < h \leq 1/2$.

Proof: Let τ_i denote the ith eigenvalue of $P(d)$ corresponding to λ_i of $L(d)$. Then we have $\tau_i = 1 - h\lambda_i$, $i = 1, 2, \ldots, N$. From Lemma 5, one can check that $\tau_i \in (-1, 1]$ for all $i = 1, 2, \ldots, N$ if and only if $0 < h < 1$ for even N and $0 < h \leq 1$ for odd N. Using similar arguments as those in the proof of Theorem 1, one can show that

$$\lim_{k\to\infty} y(kh) = d. \tag{26}$$

So we have proven the first statement of the theorem.

Now we prove the second statement of the theorem. For sufficiency, one can check that when $0 < h \leq 1/2$, all the entries of matrix $P(d)$ are nonnegative because $d_i > 0$. Moreover, no row of $P(d)$ has only zero entries. Since $y(0) > 0$, the solution to system (24) satisfies $y(kh) > 0$ for all $k = 0, 1, 2, \ldots$ For necessity, we consider the case when $h > 1/2$. Then one can construct a circle formation characterized by such a d that $1 - h(d_2/(d_2 + d_1) + d_N/(d_1 + d_N)) < 0$. Now check the first element in the vector y

$$y_1(h) = \left[1 - h\left(\frac{d_2}{d_2 + d_1} + \frac{d_N}{d_1 + d_N}\right)\right]y_1(0)$$
$$+ \frac{hd_1}{d_2 + d_1}y_2(0) + \frac{hd_1}{d_1 + d_N}y_N(0).$$

So there must exist a vector $y(0)$ satisfying $y(0) > 0$ and $\sum_{i=1}^{N} y_i(0) = 2\pi$ such that $y_1(h) \leq 0$, which implies that the order preservation property is violated. ■

We further consider the Uniform Circle Formation Problem.

Corollary 2: The Uniform Circle Formation Problem is solved with order preservation under the sampled-data control law

$$u_i(t) = \frac{y_i(kh) - y_{i^-}(kh)}{2}$$
$$t \in [kh, kh+h),\ k = 0, 1, 2, \ldots;\ i = 1, 2, \ldots, N \tag{27}$$

if and only if $0 < h < 1$ for even N and $0 < h \leq 1$ for odd N.

The proof is similar to that of Theorem 2. To save space, here we omit it.

In the next section, we consider another scenario that may arise in real applications where the formations need to be generated within finite time.

V. Finite-Time Control Law

The control laws discussed in the previous two sections can only guarantee that the circle formation will be formed as time goes to infinity. In some real applications, finite-time convergence is required. In this section, we try to solve the Finite-time Circle Formation Problem by using the technique of adopting time-varying gains.

A. Control Without Sampled Data

Consider the following control law:

$$u_i(t) = \kappa(t)\left[\frac{d_{i-}}{d_i + d_{i-}} y_i(t) - \frac{d_i}{d_i + d_{i-}} y_{i-}(t)\right] \quad i = 1, 2, \ldots, N, \tag{28}$$

where $\kappa(t)$ is a time-varying feedback gain to be designed. The corresponding overall closed-loop system becomes

$$\dot{y}(t) = -\kappa(t) L(d) y(t). \tag{29}$$

From Lemma 5, we know that there exists a nonsingular matrix $X \in \mathcal{M}_n$ such that

$$X^{-1} L(d) X = \operatorname{diag}\{0, \lambda_2, \ldots, \lambda_N\}. \tag{30}$$

Let $\eta = X^{-1} y$, and then we have

$$\dot{\eta}(t) = -\kappa(t) \operatorname{diag}\{0, \lambda_2, \ldots, \lambda_N\} \eta(t). \tag{31}$$

The main result of this subsection is as follows.

Theorem 3: Given any finite time t_f, the time-varying feedback control law (28) with

$$\kappa(t) = \frac{c}{t_f - t}, \qquad t \in [0, t_f) \tag{32}$$

solves the Finite-time Circle Formation Problem with order preservation as time approaches t_f, where c is a positive constant.

To prove this theorem, we will need the following lemma.

Lemma 6: The control law (28) solves the Finite-time Circle Formation Problem as time approaches t_f if $\eta_i(t)$ goes to 0 as $t \to t_f$, $i = 2, 3, \ldots, N$.

Proof: Without loss of generality, we assume that the first column of the matrix X is d. Then we have

$$\lim_{t \to t_f} y(t) = X \lim_{t \to t_f} \eta(t) = X[\eta_1(0), 0, \ldots, 0]^T = \eta_1(0) d.$$

Since $\sum_{i=1}^N y_i = 2\pi$ and $\sum_{i=1}^N d_i = 2\pi$, it must be true that $\lim_{t \to t_f} y(t) = d$. ■

Proof of Theorem 3: From (31), we have

$$\dot{\eta}_i(t) = \frac{-c\lambda_i}{t_f - t} \eta_i(t), \qquad t \in [0, t_f) \tag{33}$$

which implies that

$$\eta_i(t) = \left(\frac{t_f - t}{t_f}\right)^{c\lambda_i} \eta_i(0), \qquad t \in [0, t_f). \tag{34}$$

Since c and λ_i are positive, it follows that

$$\eta_i(t) \text{ goes to } 0, \ \text{ as } t \to t_f, \ i = 2, 3, \ldots, N. \tag{35}$$

From Lemma 6, we know that the control law in the form of (28) and (32) solves the Finite-time Circle Formation Problem as time approaches t_f.

The solution of y is

$$y(t) = \exp\left[c \ln \frac{t_f - t}{t_f} L(d)\right] y(0). \tag{36}$$

From Theorem 1, we know that $e^{-L(d)} > 0$. Since $c > 0$ and $t \in [0, t_f)$, it must be true that $\exp[c \ln((t_f - t)/t_f) L(d)] > 0$ and thus $y(t) > 0$. ■

Remark 1: In the proof of Theorem 3, from (34) we have

$$\dot{\eta}_i(t) = c\lambda_i \frac{(t_f - t)^{c\lambda_i - 1}}{t_f^{c\lambda_i}} \eta_i(0). \tag{37}$$

If we pick the value of c in such a way that $c\lambda_i > 1$, then $\dot{\eta}_i(t)$ is bounded for all $t \in [0, t_f)$. It follows then that $u_i(t)$ is bounded for all $t \in [0, t_f)$. Said differently, if we pick a sufficiently large c, the control input in the form of (28) and (32) is bounded.

Next we expand Theorem 3 further to include more general forms of control inputs.

Theorem 4: Given any finite time t_f, the time-varying feedback control law (28) with

$$\kappa(t) = \dot{K}(t) \tag{38}$$

solves the Finite-time Circle Formation Problem with order preservation as time approaches t_f, if the signal $K(t)$ satisfies $\dot{K}(t) > 0$ for $t \in [0, t_f)$ and $K(t) \to +\infty$ as $t \to t_f$. Furthermore, the corresponding input is always bounded if $\dot{K}(t) \exp[-K(t)]$ is bounded for $t \in [0, t_f)$.

Proof: The finite-time convergence and the boundedness of the input follow directly from the fact that

$$\eta_i(t) = \exp[\lambda_i(K(0) - K(t))] \eta_i(0). \tag{39}$$

To prove order preservation, we exam the solution of y

$$y(t) = \exp[(K(0) - K(t)) L(d)] y(0). \tag{40}$$

Since $\dot{K}(t) > 0$, we have $K(0) - K(t) < 0$. Thus, similar to the proof of Theorem 1, we have $\exp[(K(0) - K(t)) L(d)]$ is positive, which implies that $y(t) > 0$ for all $t \in [0, t_f)$. ■

B. Control With Sampled Data

The control law considered in this subsection has a similar form compared with that in the previous subsection:

$$u_i(t) = \psi(k)\left[\frac{d_{i-}}{d_i + d_{i-}} y_i(kh) - \frac{d_i}{d_i + d_{i-}} y_{i-}(kh)\right], \quad \text{for } t \in [kh, kh+h), \ k = 0, 1, 2, \ldots, N-2, \ i = 1, 2, \ldots, N, \tag{41}$$

where $h = t_f/(N-1)$, and $\psi(0), \psi(1), \ldots, \psi(N-2)$ is a sequence of time-varying feedback gains to be designed. The corresponding overall system becomes

$$y(kh+h) = H(k,d)y(kh), \quad k = 0,1,2,\ldots,N-2 \tag{42}$$

where

$$H(k,d) = I_N - \psi(k)hL(d). \tag{43}$$

The main result of this subsection is as follows.

Theorem 5: The time-varying feedback control law (41) with

$$\psi(k) = \frac{1}{h\lambda_{N-k}}, \quad k = 0,1,\ldots,N-2 \tag{44}$$

solves the Finite-time Circle Formation Problem when $t = t_f$.

Proof: Let $\sigma_{i,k}$ be the ith eigenvalue of $H(k,d)$ corresponding to λ_i of $L(d)$, and then we have $\sigma_{i,k} = 1 - \psi(k)h\lambda_i$, $i = 1,\ldots,N$. From Lemma 5, there exists a nonsingular matrix $X \in \mathcal{M}_n$ such that for all $k = 0,1,\ldots,N-2$,

$$X^{-1}H(k,d)X = V(k) = \mathrm{diag}\{1, \sigma_{2,k}, \ldots, \sigma_{N,k}\}. \tag{45}$$

Since $\psi(k) = 1/h\lambda_{N-k}$, we have the elements in the $(N-k)$th row of $V(k)$ are all zero. It follows that the matrix $V(N-2)\cdots V(0) = \mathrm{diag}\{1,0,\ldots,0\}$. Let $\phi = J^{-1}y$, and then we have

$$\phi(t_f) = V(N-2)\cdots V(0)\phi(0) = [\phi_1(0), 0, \ldots, 0]^T. \tag{46}$$

Similar to the argument in the the proof of Theorem 3, we have $y = d$ when $t = t_f$. ■

Note that a bit different from standard distributed control, to calculate the gains ψ in (44), we have assumed that at the control-law design stage one has the knowledge of d of the overall formation, while to implement such a control law each agent only needs to use the local information of the y_i's. Also note that in this case, we have not been able to provide a rigorous proof about order preservation, although one can see from the simulation results in the next section that the ordering of the agents are indeed preserved. While we have proved stability under our proposed finite-time control law (41) (44) with the sampling period $h = t_f/(N-1)$, where t_f is the given finite time, in practice, if $t_f/(N-1)$ is large one can pick a smaller sampling period in order to deal with possible measurement noises. In the next section, we use simulations to show the effectiveness of the proposed control laws.

VI. Simulations

To verify the effectiveness of our proposed control laws in the previous three sections, we carry out numerical simulations in this section. In Fig. 2, we show the simulation results of the way-point control laws (6) and (23) without and with sampled data respectively. In Fig. 3, we show the simulation results of the finite-time control laws (28) (32) and (41) (44) without and with sampled data, respectively.

In all those simulations, the initial angular positions of the N agents are generated randomly satisfying the initial condition (1). The desired circle formation in the simulation of Circle Formation Problem is also determined randomly. For ease of comparison, we use the same initial angular positions and desired admissible circle formation for each case, where we present the angular distance between each pair of neighboring agents, and the differences between current angular distances and the desired ones between each pair of neighboring agents.

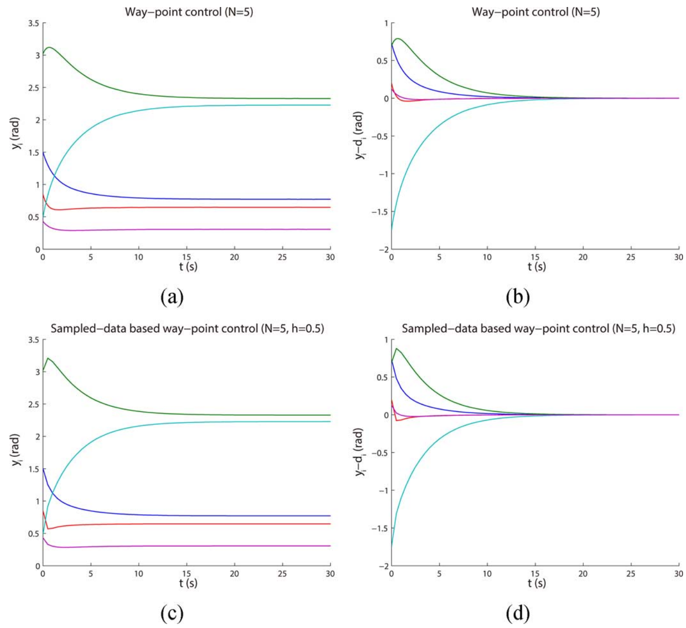

Fig. 2. Simulation results of the proposed way-point control law for the Circle Formation Problem when $N = 5$. (a)(b) the continuous-time case under control law (6); (c)(d) the sampled-data case under control law (23) with $h = 0.5$. (a)(c) angular distance between each pair of neighboring agents; (b)(d) the difference between current angular distance and the desired one between each pair of neighboring agents.

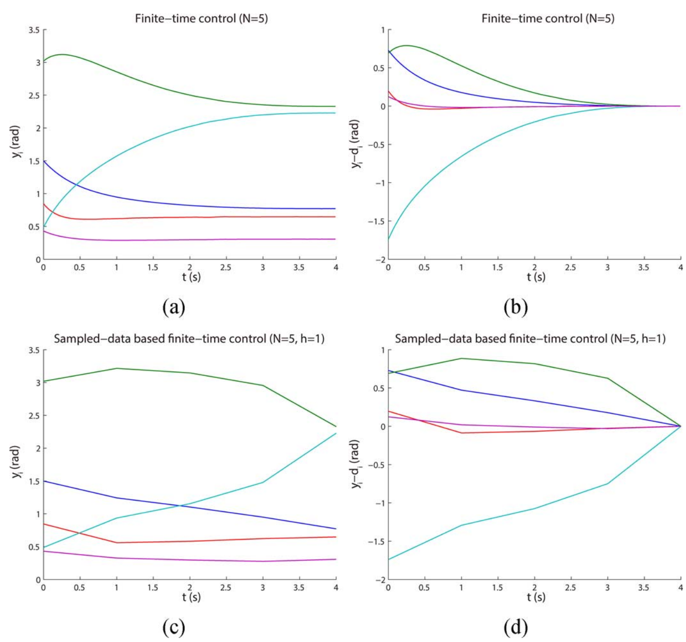

Fig. 3. Simulation results of the finite-time way-point control law for the Circle Formation Problem for a preset time $t_f = 4$ when $N = 5$. (a)(b) the continuous-time case under control law (28) (32) with $c = 10$; (c)(d) the sampled-data case under control law (41) (44) with $h = 1$. (a)(c) angular distance between each pair of neighboring agents; (b)(d) the difference between current angular distance and the desired one between each pair of neighboring agents.

The simulation results have shown that the groups of agents can converge asymptotically (resp. converge at a preset time t_f) to the desired circle formation under the way-point control law (resp. finite-time control law). In particular, the figures have demonstrated clearly that the agents preserve their orderings under our proposed control laws.

VII. Conclusion

In this paper, we have proposed distributed control laws for a group of autonomous mobile agents to realize any given circle formation. Control laws using sampled data and those guaranteeing convergence in finite time have also been studied. We have paid special attention to the property of order preservation, which can be desirable in real applications. The results provide a simple yet effective method to solve the circle-forming problem, which complements existing results. From a practical point of view, our control laws can incorporate sampled data and prevent collision between agents. Because of the linearity of the form of the control laws, they require less computation time and are thus more suitable to be implemented in real robotic systems.

However, the circle formation problem that we considered in this paper is under the assumption that the robots are initially positioned on the prescribed circle already. Although we have borrowed the assumption from the existing literature, this still leads to complementary research questions about how to design control laws to lead agents to move onto the circle when they move in the two-dimensional space of a plane. We are also interested in using robotic testbed to test the designed control strategies.

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